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Zero-sequence impedance of a three-phase transmission line with ground return

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ZERO-SEQUENCE IMPEDANCE OF A THREE-PHASE
TRANSMISSION LINE WITH GROUND RETURN

by

Marion Edwin Forsman

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Electrical Engineering

Approved:

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1954

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SYMBOLS AND UNITS

The Rationalized MKS System of Units is Used

- A = magnetic vector potential, amperes
- a, b, c = transmission line conductors
- d = conductor radius, meters
- D = dh
- E = electric intensity positive in the direction of increasing potential, volts/meter
- F(u), G(u), $\psi(u)$ = arbitrary functions of u
- g = ground or earth
- H = magnetic intensity, amperes/meter
- H_0, H_1 = Hankel functions
- h = transmission line conductor height, meters
- I = zero-sequence current, amperes
- i = current density, microamperes/meter²
- \mathcal{I} = imaginary component
- j = operator $\sqrt{-1}$
- L = inductance, henries
- m = defined function of u
- P = power, watts
- R = resistance, ohms
- r_e = effective resistance of the earth, ohms

SYMBOLS AND UNITS (continued)

R	= real component
S	= as
s	= transmission line conductor spacing, meters
t	= time, seconds
u, v	= variables of integration
V	= scalar electric potential, volts
X	= ax
X_0	= effective reactance of the earth, ohms
x, y, z	= rectangular coordinates, meters
Y	= ay
Z	= internal impedance of conductor, ohms
Z	= impedance, ohms
α^2	= $\omega\mu\sigma$
γ	= attenuation constant, per meter
δ, β	= real positive constants
ϵ	= dielectric constant of free space = $8.854(10^{-12}) = \frac{10^{-9}}{36\pi}$, farad/meter
σ	= conductivity of the earth, mhos/meter
ω	= angular frequency, radians/second
μ	= permeability of free space = $4\pi(10^{-7})$ henry/meter
ρ	= radial distance from transmission line, meters

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VITA

Marion Edwin Forsman was born July 19, 1912 at Raymondville, Texas, the son of Gyron Benjamin Forsman and Mary Esther Cheney. He received his elementary education at San Benito, Texas; his high-school education at San Benito, Texas and Grand Junction, Colorado. He was graduated from San Benito High School in 1929. He was employed in various businesses in Texas, Louisiana, and Colorado before he entered The University of Texas as a freshman in 1935. He received the degrees of Bachelor of Science in Electrical Engineering and Master of Science in Electrical Engineering in June 1940 at The University of Texas. Professor H. W. Warner was in charge of his graduate work.

He was employed as a field engineer in charge of electrical construction for various projects in Texas, Mexico, and Oklahoma before he was called to active duty in the United States Naval Reserve in 1943. On his release from active duty in 1946, he was employed as an electronics research analyst and supervisor in California. In 1947 he moved to Louisiana where he became Associate Professor of Electrical Engineering at Tulane University of Louisiana. He accepted the position of half-time Instructor in Electrical Engineering at Iowa State College in 1951 where he completed all of the requirements for the Doctor of Philosophy except the presentation of the thesis and the

VITA (continued)

final examination prior to accepting employment with the General Electric Company, Richland, Washington in 1952.

He married Dorothy Florence Kahler in Austin, Texas in 1938 and is the father of four children. He holds the commission of Lieutenant Commander in the United States Naval Reserve and is a member of the Research Reserve Company 13-2. He is a member of Tau Beta Pi, Eta Kappa Nu, Pi Mu Epsilon, American Institute of Electrical Engineers, and American Society for Engineering Education. He is an Associate Member of the Institute of Radio Engineers.

I. INTRODUCTION

The invention of the alternating current generator and transformer during the 19th century has made possible the transmission of electric power over large distances. Although the first alternating current systems were small and widely separated, the rapid growth and expansion of industry soon brought about the interconnection of the various systems until today a single interconnected polyphase system may span several states.

The problem of maintaining service during the early days consisted mainly of keeping a generator and the associated equipment in operating order. Large variations in frequency and voltage were often tolerated and system outages were not unusual. When industries began to demand more reliable service as well as more power and the expanding electrical power transmission facilities required more than just wires on poles, calculation of voltage regulation, short circuit impedance, and stability of the interconnected systems became important. As unbalanced system voltages began to receive attention, Fortescue¹ successfully applied symmetrical coordinates to the solution of unbalanced polyphase networks.

Open-wire communication lines paralleled many of the power transmission lines and interference to communications during abnormal conditions on the power system became a problem. During his studies,

Carson² derived equations for the mutual impedance between two wires with a common return path in the earth as well as the equation for the impedance of a single wire line with earth return.

Application of symmetrical coordinates to transmission line stability studies of faulted lines introduced the need for a knowledge of the impedance of the earth return path. This need was fulfilled by application of Carson's equations to the calculation of the zero-sequence impedance of a three-phase transmission line with grounded neutral. Clem³ published tables and charts to facilitate these calculations and other authors^{4, 5} have utilized Carson's results to determine transmission line impedance.

Although Carson's² equations for calculating the impedance of a three-phase transmission line with ground return have been in use for many years, there is no evidence that the impedance has been determined by a different analytical method using the same assumptions.

This thesis is a zero-sequence impedance determination obtained by calculation of the flux linkages and power losses established by the current flow in the earth beneath a three-phase transmission line with ground return and a particular configuration. While the results obtained did not lead to a method of calculating impedance which is easily applied to all problems, they did confirm the results obtained by the application of Carson's equations to the same transmission line.

II. DERIVATION OF EQUATIONS

A. The Wave Equation

The determination of the zero-sequence impedance of a transmission line is dependent on the configuration of the line. A general configuration would assume each conductor at a different height above the earth and no two equal spacings between conductors. Since the general configuration is not often used, the derivation of equations will be made for a less general case. Figure 1 is a cross-section of the transmission line which will be used for the derivation.

The following assumptions were made:

1. The earth is an homogeneous conductor of conductivity σ
2. The permeability, μ , of the earth is equal to that of free space and the capacitance is not greater than 30 times that of free space
3. In the earth all current flow is parallel to the axis of the transmission line
4. All voltages and currents are sinusoidal and contain the exponential $e^{(j\omega t - \gamma z)}$
5. The transmission line is perfectly transposed
6. The free charge density in the earth is zero.

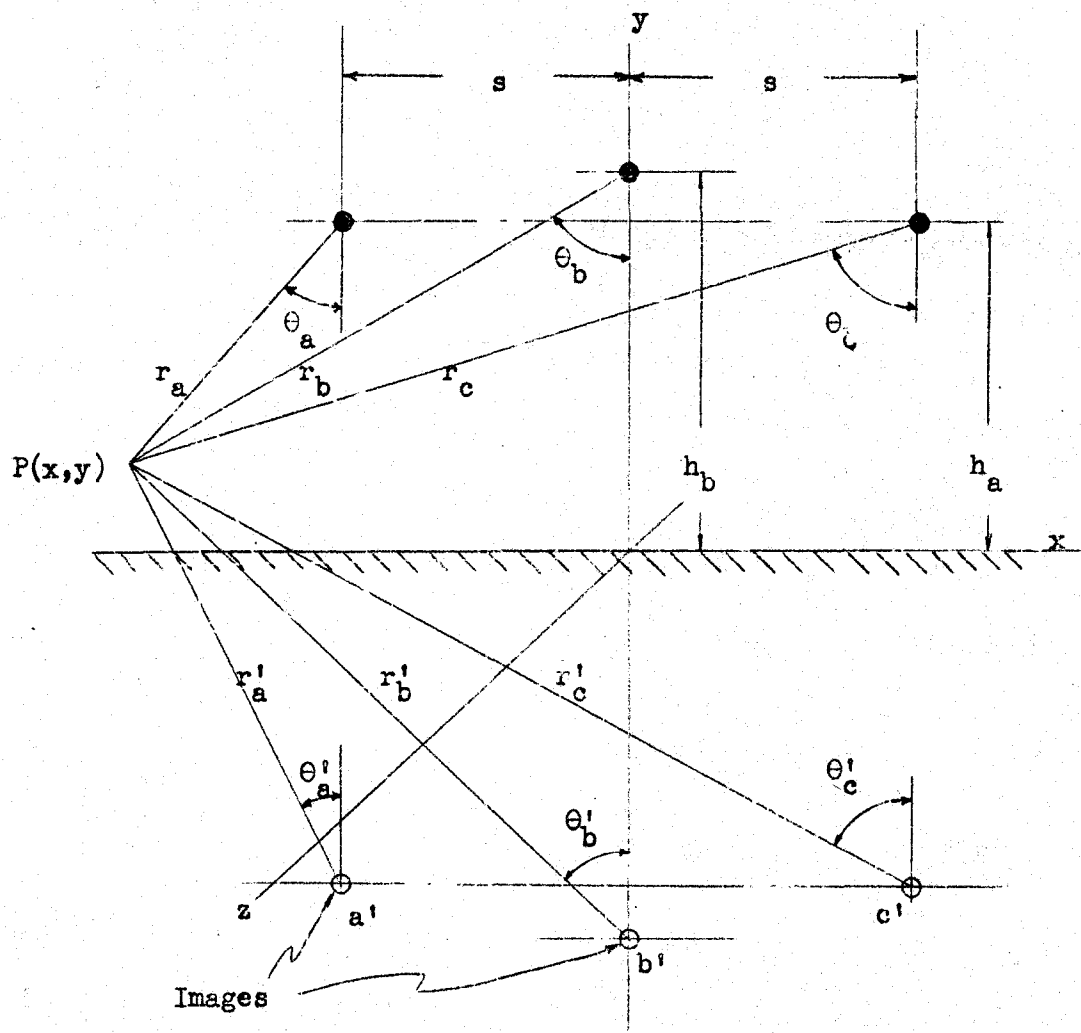


Figure 1. Transmission line conductor spacing and location of images.

The wave equation for the electric intensity is⁶:

$$\nabla^2 E = -j\omega\mu(-\sigma + j\omega\epsilon)E \quad (1)$$

The symbols are defined in the "Symbols and Units".

From assumption 3, $E_x = E_y = 0$ and from assumption 4, $E_z = |E_z|_e (j\omega t - \gamma z)$.

Thus:

$$\nabla^2 E = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \quad (2)$$

Since z is constant,

$$\nabla^2 E = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} \quad (3)$$

Substitution of equation (3) into equation (1) yields

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = -j\omega\mu(-\sigma + j\omega\epsilon)E_z \quad (4)$$

If $\frac{\omega\epsilon}{\sigma} < 0.01$, the term $j\omega\epsilon$ may be neglected. Assuming the minimum conductivity of the earth to be 10 $\mu\text{mhos/meter}$ and the frequency to be 60 cps,

$$\frac{\omega\epsilon}{\sigma} = \frac{120\pi(10^5)(30)}{36\pi(10^9)} = 0.01 \quad (5)$$

Therefore the wave equation may be written

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = j\omega\mu\sigma E_z \quad (6)$$

for the assumed conditions.

B. Derivation of the Equation for the Axial Electric Intensity in the Ground

A general solution of equation (6) with the electric field symmetrical about the x-axis may be obtained from the infinite integral⁷

$$E_z = - \int_0^{\infty} F(u) \cos(xu) e^y \psi(u) du \quad F(u) \text{ is finite} \quad (7)$$

$\psi(u)$ is finite and positive

$$y \leq 0 .$$

Substituting (7) into (6),

$$\begin{aligned} & \int_0^{\infty} u^2 F(u) \cos(xu) e^y \psi(u) du \\ & - \int_0^{\infty} F(u) \cos(xu) [\psi(u)]^2 e^y \psi(u) du \\ & + j\omega\mu\sigma \int_0^{\infty} F(u) \cos(xu) e^y \psi(u) du = 0 \end{aligned} \quad (8)$$

or

$$\int_0^{\infty} F(u) \cos(xu) e^y \psi(u) [u^2 - \psi(u)^2 + j\omega\mu\sigma] du = 0 . \quad (9)$$

Since equation (9) is identically zero for all values of x, and y,

$$\psi(u) = \sqrt{u^2 + j\omega\mu\sigma} . \quad (10)$$

Therefore

$$E_z = - \int_0^{\infty} F(u) \cos(xu) e^y \sqrt{u^2 + j\omega\mu\sigma} du . \quad (11)$$

Since

$$\nabla \times E = j \frac{\partial E_z}{\partial u} - j \frac{\partial E_z}{\partial x} = -u \left(j \frac{\partial H_x}{\partial t} + j \frac{\partial H_y}{\partial t} \right) , \quad (12)$$

$$\left. \begin{aligned} \frac{\partial E_z}{\partial y} &= -j\omega\mu H_x \\ \frac{\partial E_z}{\partial x} &= j\omega\mu H_y \end{aligned} \right\} \quad (13)$$

Substituting equation (11) into equation (13) gives

$$H_x = \frac{1}{j\omega\mu} \int_0^\infty F(u) \sqrt{u^2 + j\omega\mu\sigma} \cos(xu) e^{y\sqrt{u^2 + j\omega\mu\sigma}} du \quad (14)$$

and

$$H_y = \frac{1}{j\omega\mu} \int_0^\infty uF(u)\sin(xu) e^{y\sqrt{u^2 + j\omega\mu\sigma}} du \quad (15)$$

In the dielectric above the surface of the earth, the magnetic intensities may be written

$$H_x = H_{xa} + H_{xb} + H_{xc} + H_{xg} \quad (16)$$

$$H_y = H_{ya} + H_{yb} + H_{yc} + H_{yg} \quad (17)$$

where the subscripts a, b, and c indicate the magnetic intensities due to the conductors a, b, and c as shown in Figure 1 and the subscript g indicates the intensity due to the current in the earth.

Assuming that the current flow in the conductors has a radial symmetry, that the total current, I, divides equally between conductors, i.e., applying assumption 6, and recognizing that $\sigma = 0$ in the dielectric, the magnetic intensities due to current flowing in the wires may be written

$$\left. \begin{aligned} H_{xn} &= \frac{I \cos \theta_n}{6\pi r_n} \\ H_{yn} &= \frac{I \sin \theta_n}{6\pi r_n} \end{aligned} \right\} \quad n = a, b, c \quad (18)$$

where (see Figure 1)

$$\left. \begin{aligned} r_a &= \sqrt{(x+s)^2 + (y-h_a)^2} \\ r_b &= \sqrt{x^2 + (y-h_b)^2} \\ r_c &= \sqrt{(x-s)^2 + (y-h_a)^2} \end{aligned} \right\} \quad (19)$$

and

$$\left. \begin{aligned} \cos \theta_n &= \frac{h_n - y}{r_n} \\ \sin \theta_a &= \frac{x + s_a}{r_a} \\ \sin \theta_b &= \frac{x}{r_b} \\ \sin \theta_c &= \frac{x - s_c}{r_c} \end{aligned} \right\} \quad (20)$$

The magnetic field intensity in the dielectric due to the current in the earth may be obtained from the infinite integral⁸

$$H_{xg} = + \int_0^{\infty} G(u) \cos(xu) e^{-yu} du \quad y \geq 0 \quad (21)$$

$$H_{yg} = - \int_0^{\infty} G(u) \sin(xu) e^{-yu} du \quad y \geq 0 \quad (22)$$

At the surface of the earth where $y = 0$, the magnetic intensities may be written⁹

$$\left. \begin{aligned}
 H_{xa} &= \frac{I}{6\pi} \int_0^{\infty} e^{-h_a u} \cos(x+s)u \, du \\
 H_{xb} &= \frac{I}{6\pi} \int_0^{\infty} e^{-h_b u} \cos(xu) \, du \\
 H_{xc} &= \frac{I}{6\pi} \int_0^{\infty} e^{-h_a u} \cos(x-s)u \, du
 \end{aligned} \right\} \quad (23)$$

and

$$\left. \begin{aligned}
 H_{ya} &= \frac{I}{6\pi} \int_0^{\infty} e^{-h_a u} \sin(x+s)u \, du \\
 H_{yb} &= \frac{I}{6\pi} \int_0^{\infty} e^{-h_b u} \sin(xu) \, du \\
 H_{yc} &= \frac{I}{6\pi} \int_0^{\infty} e^{-h_a u} \sin(x-s)u \, du
 \end{aligned} \right\} \quad (24)$$

Combining equations (21) and (22) for $y = 0$ with equations (23) and (24) respectively, yields

$$H_x = \int_0^{\infty} \left\{ G(u) \cos(xu) + \frac{I}{6\pi} \left[e^{-h_a u} \cos(x+s)u + e^{-h_b u} \cos(xu) + e^{-h_a u} \cos(x-s)u \right] \right\} du \quad (25)$$

$$H_y = \int_0^{\infty} \left\{ -G(u) \sin(xu) + \frac{I}{6\pi} \left[e^{-h_a u} \sin(x+s)u + e^{-h_b u} \sin(xu) + e^{-h_a u} \sin(x-s)u \right] \right\} du \quad (26)$$

From assumption 2, the normal and tangential components of magnetic intensity are continuous across the boundary at the surface of the earth. Therefore, the values of H_x and H_y of equations (14)

and (15), with $y = 0$, are equal to H_x and H_y of equations (25) and (26) respectively. Thus

$$\begin{aligned} & \frac{1}{j\omega u} \sqrt{u^2 + j\omega\mu\sigma} F(u) \cos(xu) - G(u) \cos(xu) \\ & - \frac{I}{6\pi} \left[e^{-ha u} \cos(x+s)u + e^{-hb u} \cos(xu) \right. \\ & \left. + e^{-ha u} \cos(x-s)u \right] = 0 \end{aligned} \quad (27)$$

and

$$\begin{aligned} & \frac{u}{j\omega u} F(u) \sin(xu) + G(u) \sin(xu) - \frac{I}{6\pi} \left[e^{-ha u} \sin(x+s)u \right. \\ & \left. + e^{-hb u} \sin(xu) + e^{-ha u} \sin(x-s)u \right] = 0 \end{aligned} \quad (28)$$

Simultaneous solution of equations (27) and (28) gives

$$\begin{aligned} F(u) = & \frac{I}{6\pi} \frac{j\omega\mu}{\sqrt{u^2 + j\omega\mu\sigma} + u} \left\{ \frac{1}{\cos(xu)} \left[e^{-ha u} \cos(x+s)u \right. \right. \\ & \left. \left. + e^{-hb u} \cos(xu) + e^{-ha u} \cos(x-s)u \right] \right. \\ & \left. + \frac{1}{\sin(xu)} \left[e^{-ha u} \sin(x+s)u + e^{-hb u} \sin(xu) \right. \right. \\ & \left. \left. + e^{-ha u} \sin(x-s)u \right] \right\} \end{aligned} \quad (29)$$

and

$$\begin{aligned} G(u) = & \frac{I}{6\pi} \left(1 - \frac{u}{\sqrt{u^2 + j\omega\mu\sigma} + u} \right) \frac{1}{\sin(xu)} \\ & \left[e^{-ha u} \sin(x+s)u + e^{-hb u} \sin(xu) \right. \\ & \left. + e^{-ha u} \sin(x-s)u \right] - \frac{I}{6\pi} \left(\frac{u}{\sqrt{u^2 + j\omega\mu\sigma} + u} \right) \\ & \frac{1}{\cos(xu)} \left[e^{-ha u} \cos(x+s)u + e^{-hb u} \cos(xu) \right. \\ & \left. + e^{-ha u} \cos(x-s)u \right] \end{aligned} \quad (30)$$

Substituting the value of $F(u)$ into equation (11) gives the value of E_z in the earth.

$$\begin{aligned}
 E_z = & \frac{-I}{6\pi} \int_0^{\infty} \frac{j\omega\mu e^{-y\sqrt{u^2 + j\omega\mu\sigma}} \cos(xu)}{\sqrt{u^2 + j\omega\mu\sigma} + u} \left\{ \frac{1}{\cos(xu)} \right. \\
 & \left[e^{-h_a u} \cos(x+s)u + e^{-h_b u} \cos(xu) + e^{-h_a u} \cos(x-s)u \right] \\
 & + \frac{1}{\sin(xu)} \left[e^{-h_a u} \sin(x+s)u + e^{-h_b u} \sin(xu) \right. \\
 & \left. \left. + e^{-h_a u} \sin(x-s)u \right] \right\} du . \quad (31)
 \end{aligned}$$

There are two general configurations which are in common use in three-phase transmission line construction. For one configuration $h_a < h_b$ while for the other $h_a = h_b$. For the remainder of this derivation assume that $h_a = h_b = h$. Substituting this value into equation (31) and applying the proper trigonometric identities yields

$$\begin{aligned}
 E_z = & -\frac{j\omega\mu I}{3\pi} \int_0^{\infty} \frac{e^{-(hu - y\sqrt{u^2 + j\omega\mu\sigma})} \cos(xu)}{\sqrt{u^2 + j\omega\mu\sigma} + u} \\
 & [2\cos(su) + 1] du \\
 = & -\frac{\omega\mu I}{3\pi\omega\mu\sigma} \int_0^{\infty} (\sqrt{u^2 + j\omega\mu\sigma} - u) e^{-(hu - y\sqrt{u^2 + j\omega\mu\sigma})} \\
 & \cos(xu) [2\cos(su) + 1] du . \quad (32)
 \end{aligned}$$

Let

$$\left. \begin{aligned}
 \sqrt{\omega\mu\sigma} &= a \\
 S &= as \\
 D &= ah \\
 X &= ax \\
 Y &= ay \\
 u &= av
 \end{aligned} \right\} \quad (33)$$

and substitute into equation (32). This substitution will not change the limits and therefore

$$E_z = -\frac{\omega\mu I}{3\pi} \int_0^{\infty} (\sqrt{v^2 + j} - v) e^{-(Dv - Y\sqrt{v^2 + j})} \cos(Xv) [2\cos(Sv) + 1] dv \quad Y \leq 0 \quad (32a)$$

Comparing equation (32a) with Carson's¹⁰ equation (14) reveals that the addition of two wires adds one term which is identical with Carson's equation except for the additional factor $2\cos(Sv)$.

C. Derivation of the Equation for the Electric Intensity in Air

The axial electric intensity in the dielectric may be derived from the vector and scalar potentials¹¹ as

$$E = -\frac{\partial A}{\partial t} - \nabla V \quad (34)$$

or

$$E_z = -\frac{\partial A_z}{\partial t} - \frac{\partial V}{\partial z} \quad (35)$$

when $E_x = E_y = 0$.

Now

$$E_z(x,y) - E_z(x,0) = -j\omega [A_z(x,y) - A_z(x,0)] - \frac{\partial}{\partial z} [V(x,y) - V_0] \quad (37)$$

where $E_z(x,0)$ is the axial electric intensity at the surface of the earth and the distance x from the center of the transmission line, $V(x,y) - V_0$ is the difference in potential between $P(x,y)$ and the

earth, and

$$(38) \quad \Delta_z^2(x,y) - \Delta_z^2(x,0) = \int_Y^0 H_x^2(x,y) dy \cdot$$

From equations (18), (19), (20), and (21),

$$\int_Y^0 H_x^2(x,y) dy = \int_Y^0 \int_0^\infty G(u) \cos(xu) e^{-yu} du dy$$

$$(39) \quad + \int_Y^0 \frac{G(u)}{I} \left[\frac{(x+s)^2 + (h_a - y)^2}{h_a - y} + \frac{h_b - y}{x^2 + (h_b - y)^2} \right. \\ \left. + \frac{h_a - y}{(x-s)^2 + (h_a - y)^2} \right] dy$$

$$= \int_Y^0 \int_0^\infty G(u) \cos(xu) e^{-yu} dy du$$

$$(40) \quad - \frac{I}{12\pi} \left[\frac{h_a - y}{(x+s)^2 + (h_a - y)^2} \right. \\ \left. + \frac{h_b - y}{x^2 + (h_b - y)^2} + \frac{h_a - y}{(x-s)^2 + (h_a - y)^2} \right] + \frac{h_a - y}{x^2 + (h_b - y)^2}$$

Equation (40) will now be evaluated for the particular case where $h_a = h_b = h$. Equation (30) may be written as

$$(41) \quad G(u) = \frac{I e^{-hu}}{Gh} \left[\frac{\sqrt{u^2 + h^2}}{u - \sqrt{u^2 + h^2}} + 1 \right] \cos(au) + 1$$

Thus, the double integral of equation (40) yields

$$\begin{aligned}
& \int_0^{\infty} \int_0^y G(u) \cos(xu) e^{-yu} dy du \\
&= \frac{I}{6\pi} \int_0^{\infty} \int_0^y \frac{\sqrt{u^2 + ja^2} - u}{\sqrt{u^2 + ja^2} + u} \\
&\quad [2\cos(su) + 1] e^{-hu} \cos(xu) e^{-yu} dy du \\
&= -\frac{I}{6\pi} \int_0^{\infty} \frac{\sqrt{u^2 + ja^2} - u}{\sqrt{u^2 + ja^2} + u} [2\cos(su) + 1] \\
&\quad e^{-(h+y)u} \cos(xu) \frac{du}{u} \\
&\quad + \frac{I}{6\pi} \int_0^{\infty} \frac{\sqrt{u^2 + ja^2} - u}{\sqrt{u^2 + ja^2} + u} [2\cos(su) + 1] \\
&\quad e^{-hu} \cos(xu) \frac{du}{u} . \tag{42}
\end{aligned}$$

Rationalizing equation (42) and collecting terms gives

$$\begin{aligned}
& \int_0^{\infty} \int_0^y G(u) \cos(xu) e^{-yu} dy du = \\
& -\frac{jI}{3\pi a^2} \int_0^{\infty} (\sqrt{u^2 + ja^2} - u) [2\cos(su) + 1] \\
&\quad e^{-(h+y)u} \cos(xu) du \\
& + \frac{jI}{3\pi a^2} \int_0^{\infty} (\sqrt{u^2 + ja^2} - u) [2\cos(su) + 1] \\
&\quad e^{-hu} \cos(xu) du \\
& - \frac{I}{6\pi} \int_0^{\infty} [2\cos(su) + 1] [e^{-(h+y)u} - e^{-hu}] \\
&\quad \cos(xu) \frac{du}{u} . \tag{43}
\end{aligned}$$

Since the tangential component of E is continuous across the boundary at the surface of the earth, the second term on the right hand side of equation (43) is seen to be $-\frac{jI}{\omega\mu} E_z(x,0)$ when $y = 0$ is substituted into equation (32). The final term in equation (43) may be written as the sum of three terms thus

$$\begin{aligned} & \frac{I}{6\pi} \int_0^\infty [2\cos(su) + 1] [e^{-(h+y)u} - e^{-hu}] \cos(xu) \frac{du}{u} = \\ & \frac{I}{6\pi} \int_0^\infty \cos(x+s)u [e^{-(h+y)u} - e^{-hu}] \frac{du}{u} \\ & + \int_0^\infty \cos(x-s)u [e^{-(h+y)u} - e^{-hu}] \frac{du}{u} \\ & + \int_0^\infty \cos(xu) [e^{-(h+y)u} - e^{-hu}] \frac{du}{u} . \end{aligned} \quad (44)$$

Integration¹² of equation (44) and substitution into equation (43) gives

$$\begin{aligned} & \int_0^\infty \int_0^y G(u) \cos(xu) e^{-yu} dy du = \\ & - \frac{jI}{3\pi a^2} \int_0^\infty (\sqrt{u^2 + ja^2} - u) [2\cos(su) + 1] \\ & e^{-(h+y)u} \cos(xu) du \\ & - \frac{jI}{\omega\mu} E_z(x,0) - \frac{I}{12\pi} \left[\ln \frac{(x+s)^2 + h^2}{(x+s)^2 + (h+y)^2} \right. \\ & \left. + \ln \frac{(x-s)^2 + (h^2 - y)^2}{(x-s)^2 + (h-y)^2} + \ln \frac{x^2 + h^2}{x^2 + (h-y)^2} \right] . \end{aligned} \quad (45)$$

Substituting equation (45) into equation (40) gives

$$\begin{aligned}
\int_0^{\infty} H_x(x,y) dy &= - \frac{jI}{3\pi a^2} \int_0^{\infty} (\sqrt{u^2 + ja^2} - u) \\
&\quad [2\cos(su) + 1] e^{-(h+y)u} \cos(xu) du \\
- \frac{j}{\omega\mu} E_z(x,0) &- \frac{I}{12\pi} \left[\ln \frac{(x+s)^2 + (h-y)^2}{(x+s)^2 + (h+y)^2} \right. \\
&+ \ln \frac{(x-s)^2 + (h-y)^2}{(x-s)^2 + (h+y)^2} \\
&\left. + \ln \frac{x^2 + (h-y)^2}{x^2 + (h+y)^2} \right]. \tag{46}
\end{aligned}$$

Therefore

$$\begin{aligned}
- j\omega\mu \int_0^{\infty} H_x(x,y) dy &= - \frac{\omega\mu I}{3\pi a^2} \left\{ \int_0^{\infty} (\sqrt{u^2 + ja^2} - u) \right. \\
&\quad \left. [2\cos(su) + 1] e^{-(h+y)u} \cos(xu) du \right\} - E_z(x,0) \\
- \frac{j\omega\mu I}{6\pi} \left[\ln \frac{r_a^i}{r_a} + \ln \frac{r_b^i}{r_b} + \ln \frac{r_0^i}{r_0} \right] &\tag{47}
\end{aligned}$$

where r_a^i and r_b^i are as shown in Figure 1.

The electric intensity in the dielectric may now be written from equations (37), (38), and (47) and assumption 4 as

$$\begin{aligned}
E_z(x,y) &= - \frac{\omega\mu I}{3\pi a^2} \int_0^{\infty} (\sqrt{u^2 + ja^2} - u) [2\cos(su) + 1] \\
&\quad e^{-(h+y)u} \cos(xu) du - \frac{j\omega\mu I}{6\pi} \ln \frac{r_a^i r_b^i r_0^i}{r_a r_b r_0} + \gamma V. \tag{48}
\end{aligned}$$

The electric intensity at the surface of the wires may be written as $j_n I$ where j_n is the internal impedance of the wire per meter

length and I is the current flowing in the wire. Since the tangential component of the electric intensity is continuous across the boundary, E_z from equation (48) must equal $\int_n \frac{I}{3}$ at the surface of the wire.

Therefore

$$\int_n = -\frac{\omega\mu}{\pi a^2} \int_0^\infty (\sqrt{u^2 + ja^2} - u) [2\cos(su) + 1] e^{-(h+y)u} \cos(xu) du - \frac{j\omega\mu}{2\pi} \ln \frac{r'_a r'_b r'_c}{r_a r_b r_c} + 3\gamma \frac{V}{I} \quad (49)$$

Now

$$\frac{1}{3} \frac{dI}{dz} = - (G + j\omega C) V \quad (50)$$

but

$$\begin{aligned} \frac{dI}{dz} &= \frac{d}{dz} |I| e^{(j\omega t - \gamma z)} \\ &= -\gamma I \end{aligned} \quad (51)$$

and thus

$$\frac{3V}{I} = \frac{\gamma}{G + j\omega C} \quad (52)$$

Substitution of equation (52) into equation (49) and rearranging yields

$$\begin{aligned} \gamma^2 = (G + j\omega C) \left\{ \int_n + \frac{\omega\mu}{\pi a^2} \left(\int_0^\infty (\sqrt{u^2 + ja^2} - u) \right. \right. \\ \left. \left. [2\cos(su) + 1] e^{-(h+y)u} \cos(xu) du \right) \right. \\ \left. + \frac{j\omega\mu}{2\pi} \ln \frac{r'_a r'_b r'_c}{r_a r_b r_c} \right\} \quad (53) \end{aligned}$$

The shunt admittance for each wire is the first factor and the series impedance is the second factor of equation (53). The series impedance may be written as

$$Z_n = Z_n + \frac{\omega\mu}{\pi a^2} \int_0^\infty (\sqrt{u^2 + ja^2} - u) [2\cos(su) + 1] e^{-(h+y)u} \cos(xu) du + \frac{j\omega\mu}{2\pi} \ln \frac{r'_a r'_b r'_o}{r_a r_b r_o} \quad (54)$$

The impedance of the "a" conductor will be written here as an example.

$$Z_a = Z_a + \frac{\omega\mu}{\pi a^2} \int_0^\infty (\sqrt{u^2 + ja^2} - u) [2\cos(su) + 1] e^{-2hu} \cos(xu) du + \frac{j\omega\mu}{2\pi} \ln \frac{(2h-d)[\sqrt{(s^2 + 4h^2)} - d][\sqrt{(4s^2 + 4h^2)} - d]}{d(s-d)(2s-d)} \quad (55)$$

where d = radius of conductor "a".

Since $h \gg d$ and $s \gg d$, an approximate solution may be written

$$Z_a = Z_a + \frac{\omega\mu}{\pi a^2} \int_0^\infty (\sqrt{u^2 + ja^2} - u) [2\cos(su) + 1] e^{-2hu} \cos(xu) du + \frac{j\omega\mu}{2\pi} \ln \frac{h\sqrt{(s^2 + 4h^2)}(4s^2 + 4h^2)}{ds^2} \quad (56)$$

The first and third terms of equation (56) give the impedance of the transmission line with an infinitely conducting earth while

the second term is the additional impedance due to the finite conductivity of the earth. Rewriting equation (56) and applying the proper trigonometric identities gives

$$\begin{aligned}
 Z_a = Z_a + \frac{j\omega\mu}{\pi a^2} \int_0^\infty (\sqrt{u^2 + ja^2} - u) \\
 [\cos(2su) + \cos(su) + 1] e^{-2hu} du \\
 + \frac{j\omega\mu}{2\pi} \ln \frac{h \sqrt{(s^2 + 4h^2)(4s^2 + 4h^2)}}{ds^2}
 \end{aligned} \tag{57}$$

which is identical to those derived by Carson¹³ for a single wire with mutual impedance coupling to two adjacent wires except for the factor $\frac{1}{a^2}$ which can be eliminated by using equations (33).

III. NUMERICAL SOLUTION OF THE EQUATION FOR THE ELECTRIC INTENSITY IN THE EARTH

A. IBM Machine Calculations

Equation (32a) gives the electric field intensity in the earth in integral form. A solution to this equation, when multiplied by $-\sigma$, will give the current density as a function of x , y , h , and s . A general solution for this equation was not available and attempts to solve it by series expansion were not successful. It was decided that numerical solutions for particular values of s and h would be made. Since the IBM facilities available could solve only problems involving real quantities, the integral was put into a form which contained a real term and an imaginary term.

The term $\sqrt{v^2 + j}$ is the only term in the equation which is complex and the angle associated with the term is always less than 90° . Assume that

$$\sqrt{v^2 + j} = \delta + j\beta \quad (58)$$

where δ and β are real, positive numbers.

The solution to equation (58) gives

$$\delta = \sqrt{\frac{v^2 + \sqrt{v^4 + 1}}{2}} = \frac{n}{\sqrt{2}} \quad (59)$$

$$\beta = \frac{1}{\sqrt{2(v^2 + \sqrt{v^4 + 1})}} = \frac{1}{n\sqrt{2}} \quad (60)$$

Substituting equations (59) and (60) into equation (32a) gives

$$E_z = -\frac{\omega\mu I}{3\pi} \int_0^\infty \left[\frac{1}{\sqrt{2}} \left(m + \frac{j}{m} - v \right) e^{-\left[Dv - Y \frac{1}{\sqrt{2}} \left(m + \frac{j}{m} \right) \right]} \right. \\ \left. \cos(Xv) [2\cos(Sv) + 1] dv \right] \quad (61)$$

Since $i_z = -\sigma E_z$ and since this solution is restricted to $y \leq 0$,

$$i_z(\mathcal{R}) = \frac{\omega\mu\sigma I}{3\pi} \int_0^\infty [2\cos(Sv) + 1] \\ \cos(Xv) e^{-(Dv + |Y| \frac{m}{\sqrt{2}})} \\ \left[\left(\frac{m}{\sqrt{2}} - v \right) \cos \frac{|Y|}{\sqrt{2} m} + \frac{1}{\sqrt{2} m} \sin \frac{|Y|}{\sqrt{2} m} \right] dv \quad (62)$$

and

$$i_z(\mathcal{I}) = \frac{\omega\mu\sigma I}{3\pi} \int_0^\infty [2\cos(Sv) + 1] \\ \cos(Xv) e^{-(Dv + |Y| \frac{m}{\sqrt{2}})} \\ \left[\frac{m}{\sqrt{2}} \cos \frac{|Y|}{\sqrt{2} m} - \left(\frac{m}{\sqrt{2}} - v \right) \sin \frac{|Y|}{\sqrt{2} m} \right] dv \quad (63)$$

Where

$|Y|$ = absolute value of Y

$i_z(\mathcal{R})$ = real component of current density

$i_z(\mathcal{I})$ = imaginary component of current density.

Simpson's Rule¹⁴ was used to obtain numerical solutions for a conductor height of 20 meters and three conductor spacings of 10, 20, and 40 meters. The conductivity of the earth was assumed to be 0.01 mho/meter for each case. The error¹⁵ involved in using Simpson's Rule was calculated in order to determine the proper intervals between ordinates for a given accuracy.

The IBM general purpose CPC system used for the calculations was known as the SEMAC (Series Evaluated Functions, Mixed decimal, Arithmetic Computer) and consisted of a type 605 electronic calculator and a type 412 accounting machine. The machine was capable of processing about 80 to 100 cards per minute and required between 15 and 120 minutes to evaluate equations (62) and (63) for a single point, $F(x,y)$. The SEMAC had sufficient storage capacity to obtain the solutions to both of the equations at the same time. The length of time for the evaluation depends on the rate of curvature and the rate of convergence of the integrand. The details of programming the problem on the IBM punch cards and the method of operating the machine will not be discussed.

B. Discussion

The data obtained with the IBM facilities are tabulated in Appendix C and plotted on Figures 2, 3, 4, 5, 9, 10, 11, and 12. Inspection of the data indicates that the real component of current density does not vary appreciably for the three cases studied but

that the imaginary component of current density is quite different near the transmission line and becomes approximately equal for the three cases at large distances from the line. Under these conditions, Figures 3 and 5 represent the current densities at large horizontal distances for all cases while Figure 6 is representative of the real component of current density for all cases.

Figures 6 and 7 were obtained by plotting curves, similar to Figures 2 and 3 (or 4 and 5), for constant values of x and then obtaining constant current density points from the four curves. The real values of current density are seen to decay more slowly than the imaginary values as the distance from the transmission line is increased. Figure 8 was obtained by calculating the magnitude and phase angle from the data of Appendix C and obtaining constant current density and constant phase angle points from curves similar to Figures 2 and 3.

The true accuracy of the calculations is not known. The error due to using Simpson's Rule is shown in Appendix C and for most points the error was less than three percent. The error caused by the substitution of a finite upper limit for the infinite limit in equations (52) and (53) is also not known. Calculations were carried out until the value of the ordinate calculated was less than five percent of the sum of the ordinates for small values of $|Y|$ and less than one percent for large values of $|Y|$. Errors in obtaining the value of the integrals near zero were probably quite large because of the finite value used for the upper limit.

It should be noted that the solutions may be used for other values of s , h , ω and σ provided the values of S and D are not changed and the ratio s/h is held constant.

Figure 2. Real component of zero-sequence current density under a three-phase transmission line with ten meter spacing and ground return

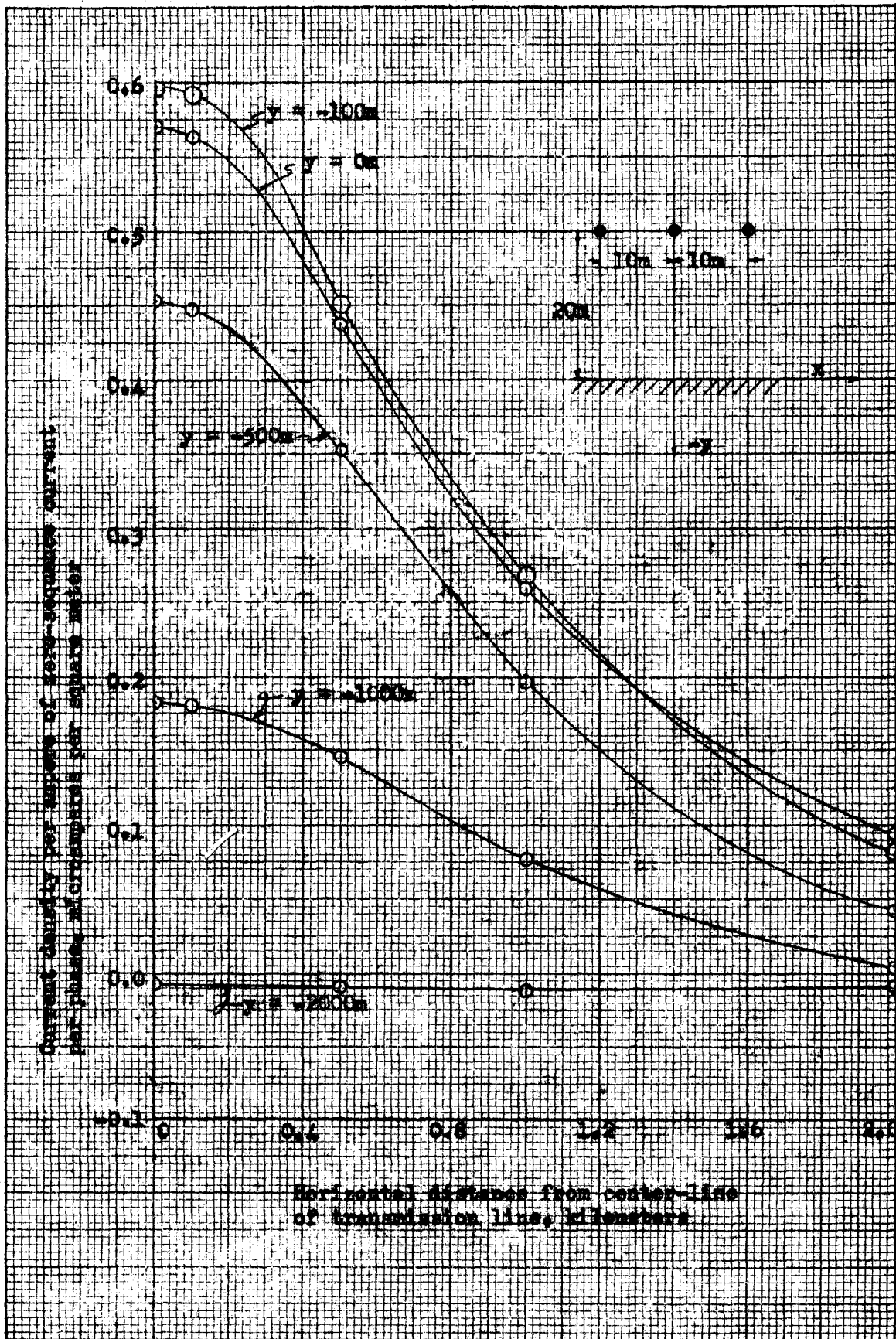


Figure 3. Continuation of Figure 2

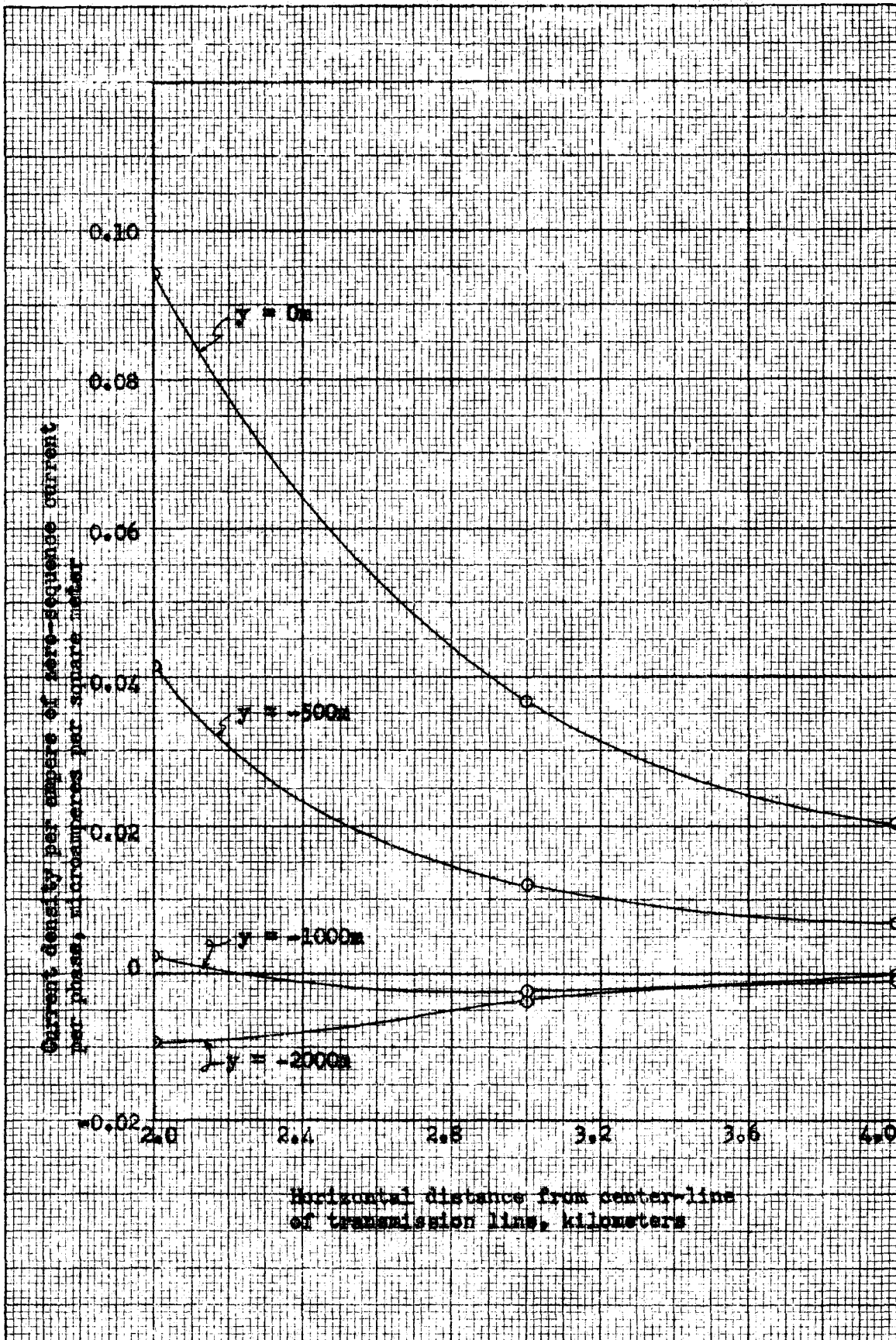


Figure 4. Imaginary component of zero-sequence current density under a three-phase transmission line with ten meter spacing and ground return

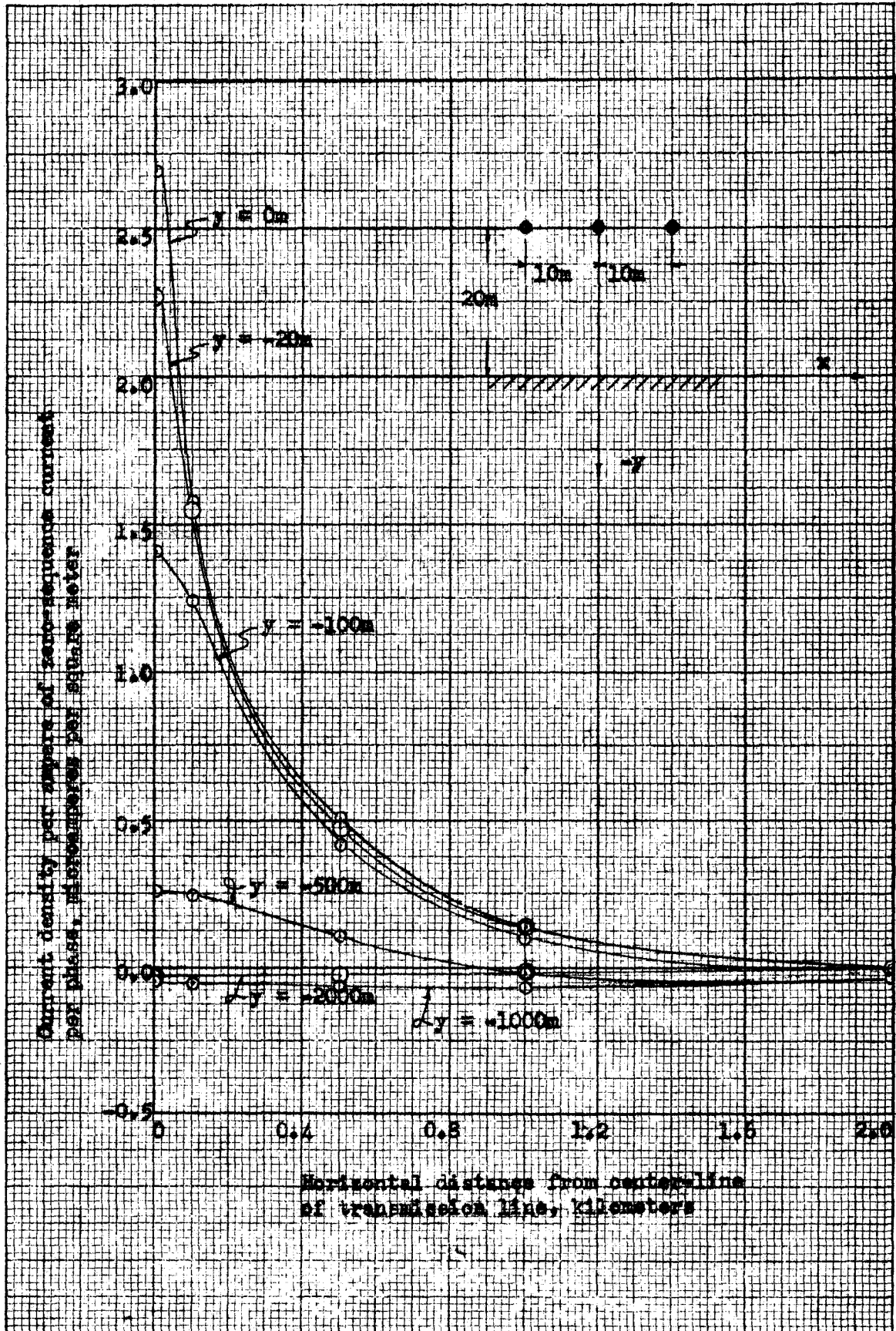


Figure 5. Continuation of Figure 4

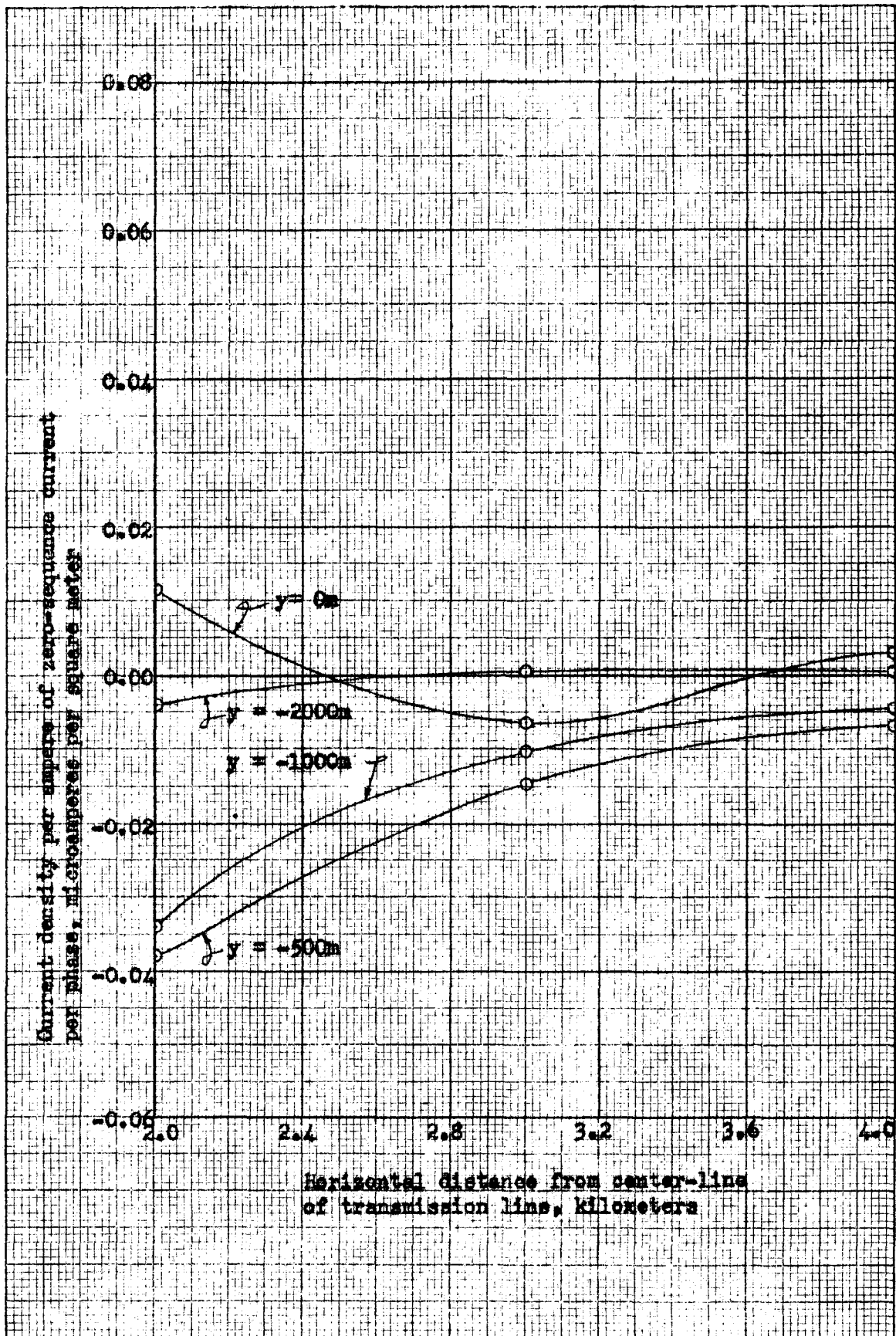


Figure 6. Constant current density plot of the real component of zero-sequence current density under a three-phase transmission line with ten meter spacing and ground return

Figure 7. Constant current density plot of the imaginary component of zero-sequence current density under a three-phase transmission line with ten meter spacing and ground return

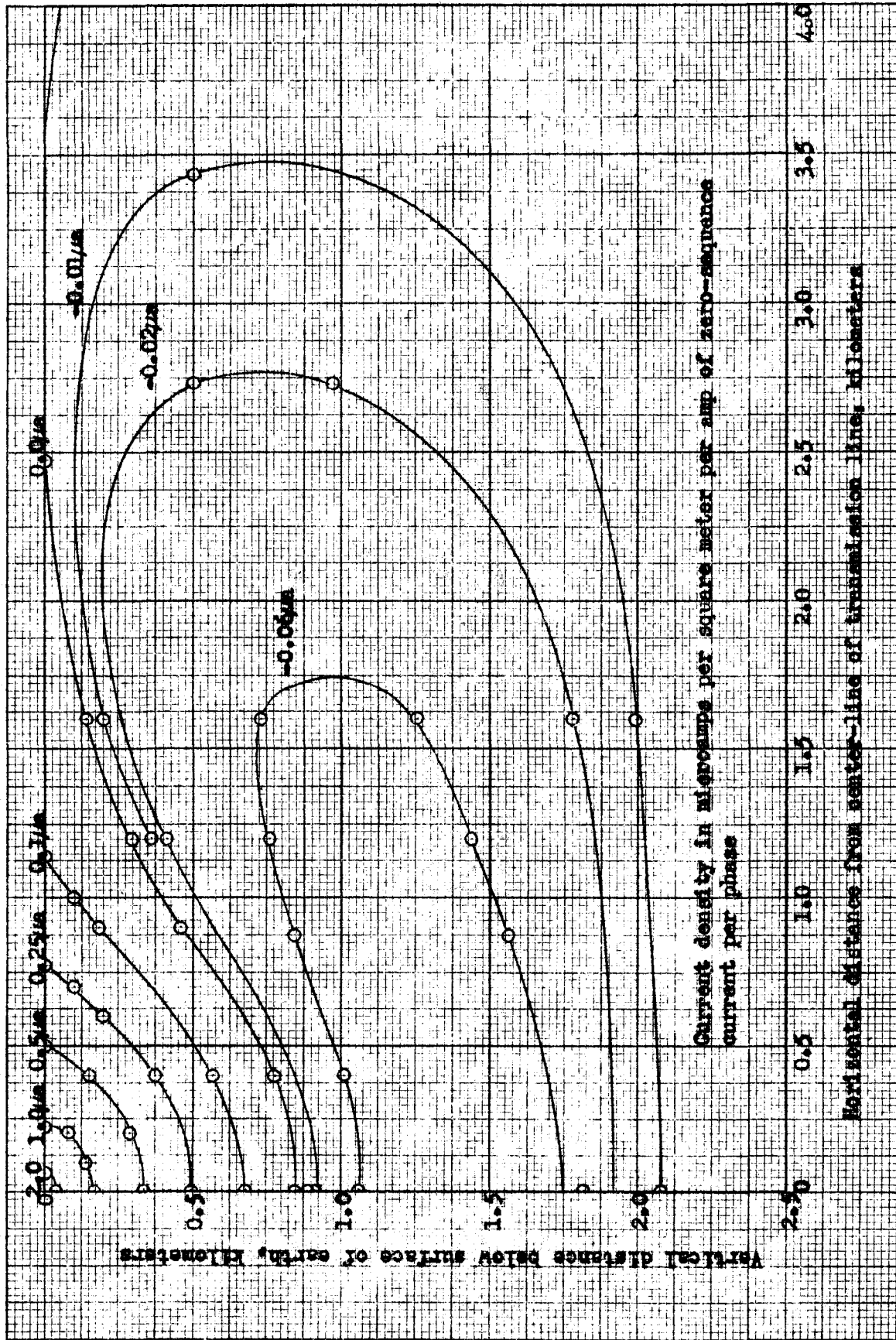


Figure 8. Constant current density plot of the magnitude and phase angle of the zero-sequence current density under a three-phase transmission line with ten meter spacing and ground return

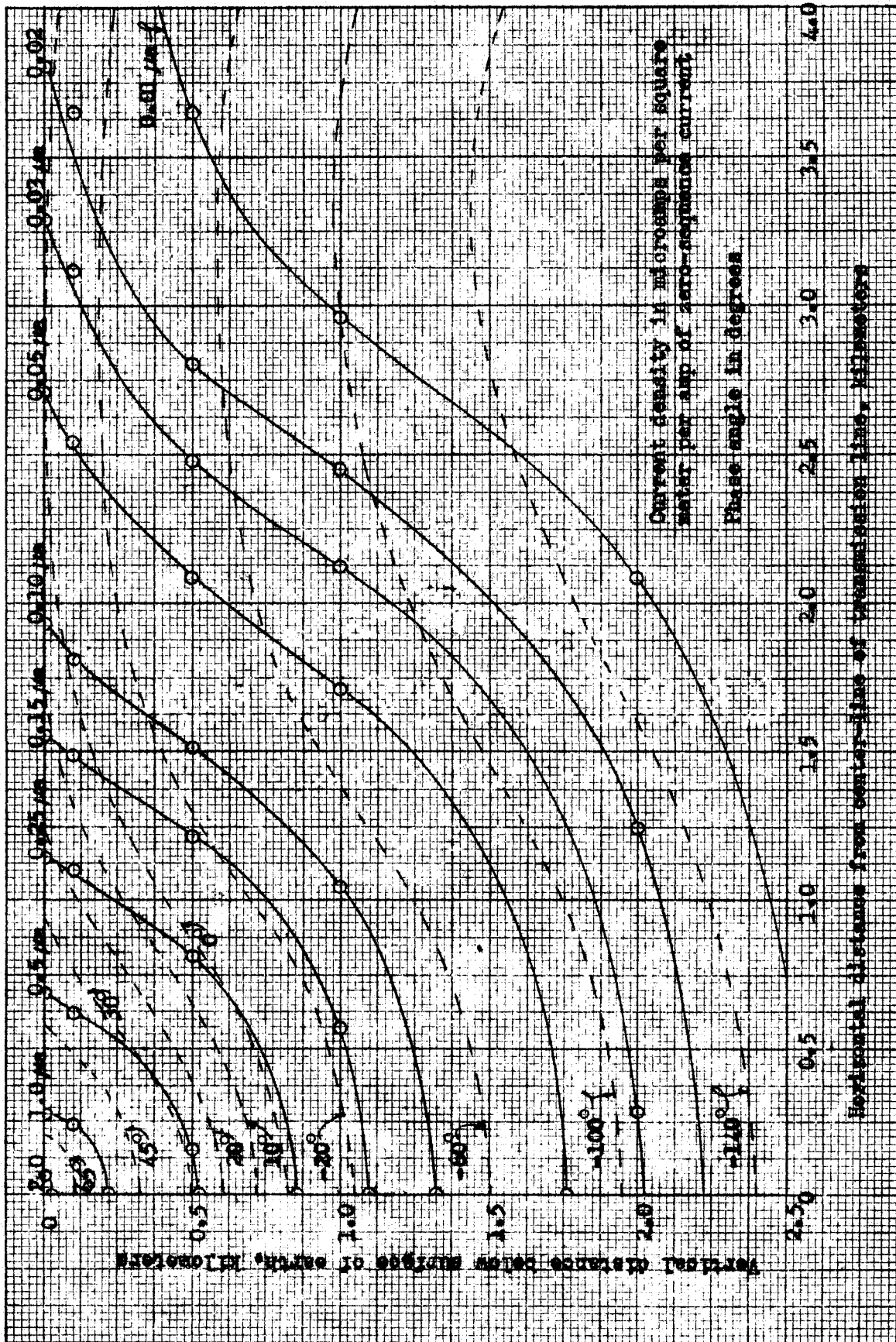


Figure 9. Real component of zero-sequence current density under a three-phase transmission line with twenty meter spacing and ground return

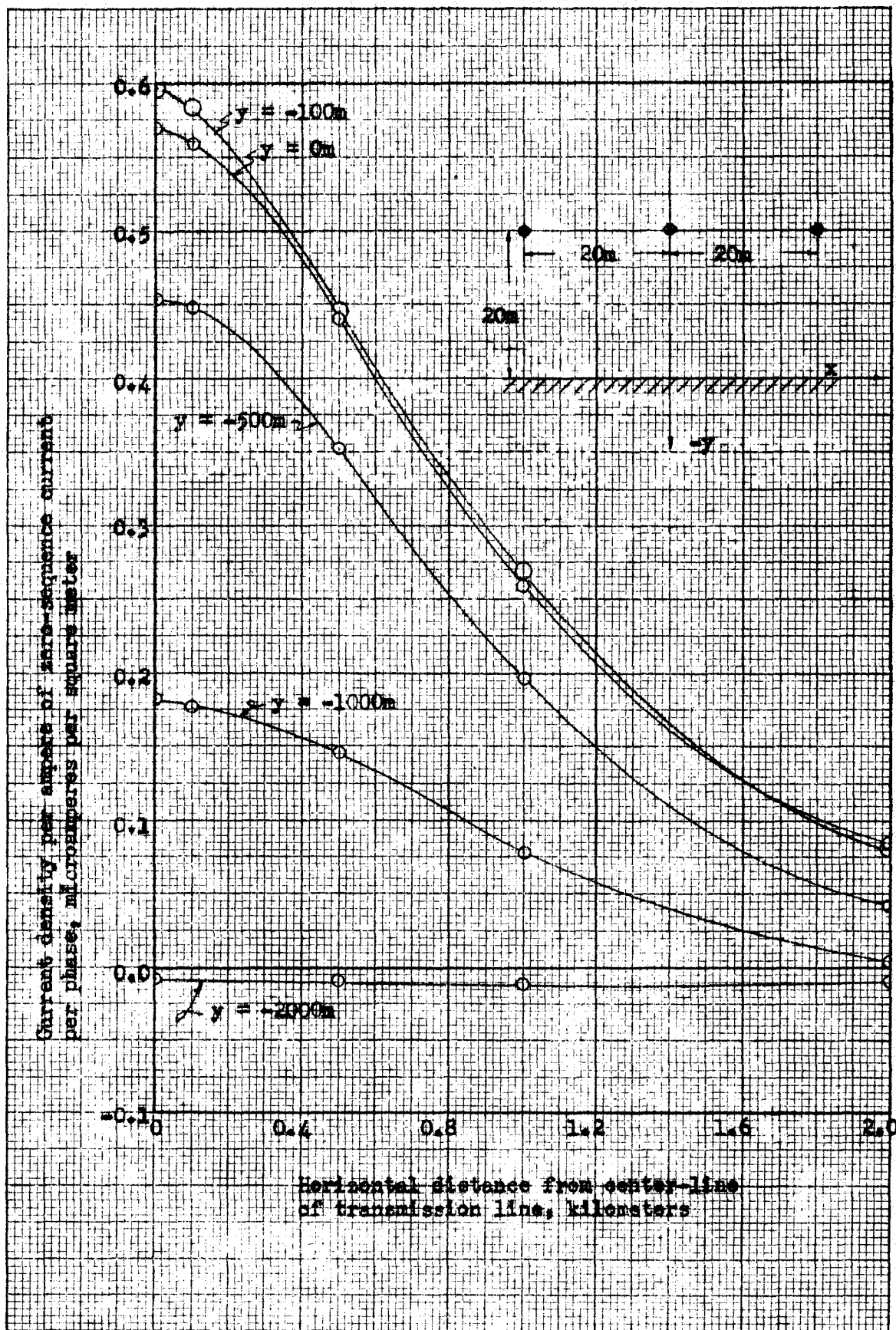


Figure 10. Imaginary component of zero-sequence current density under a three-phase transmission line with twenty meter spacing and ground return

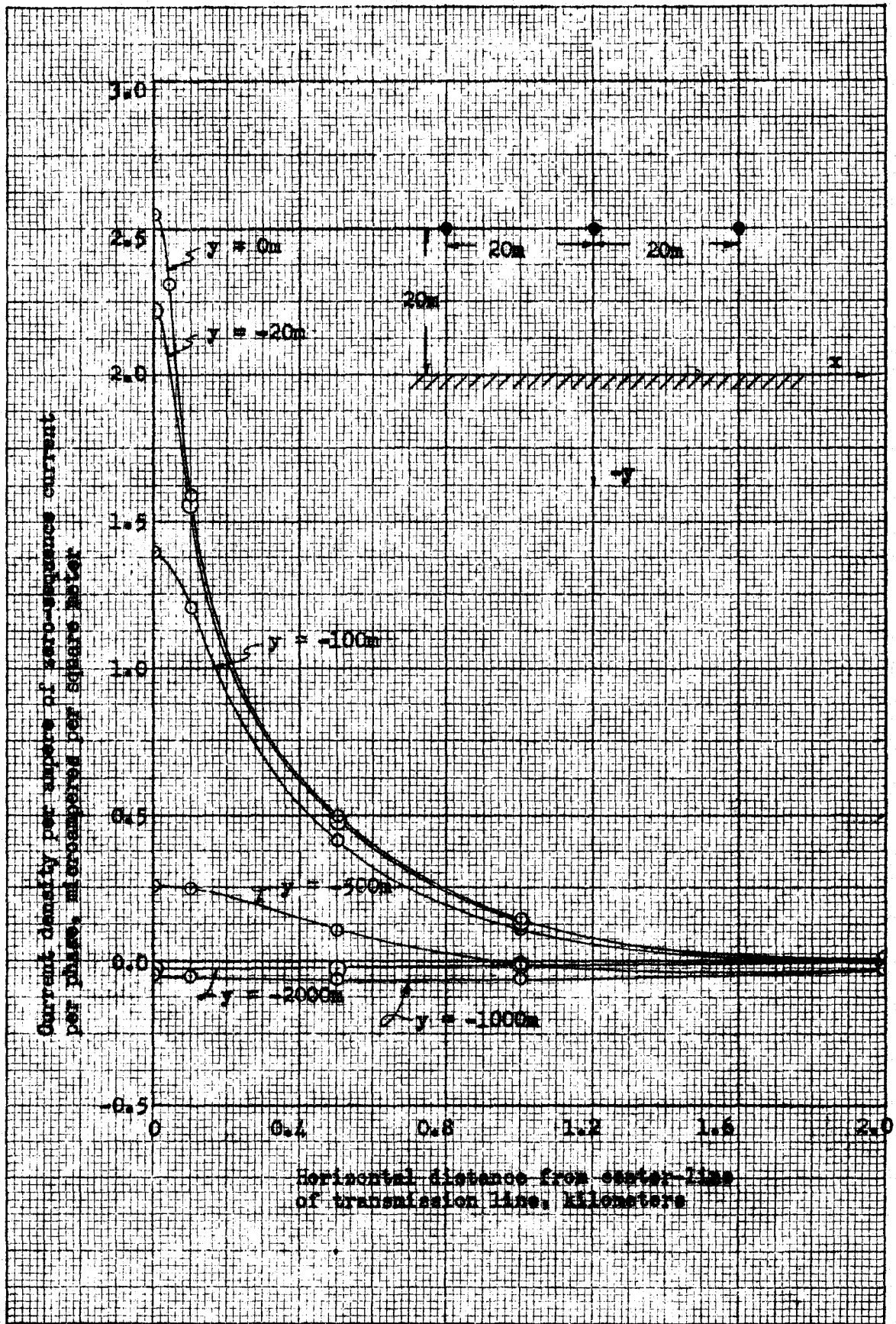


Figure 11. Real component of zero-sequence current density under a three-phase transmission line with forty meter spacing and ground return

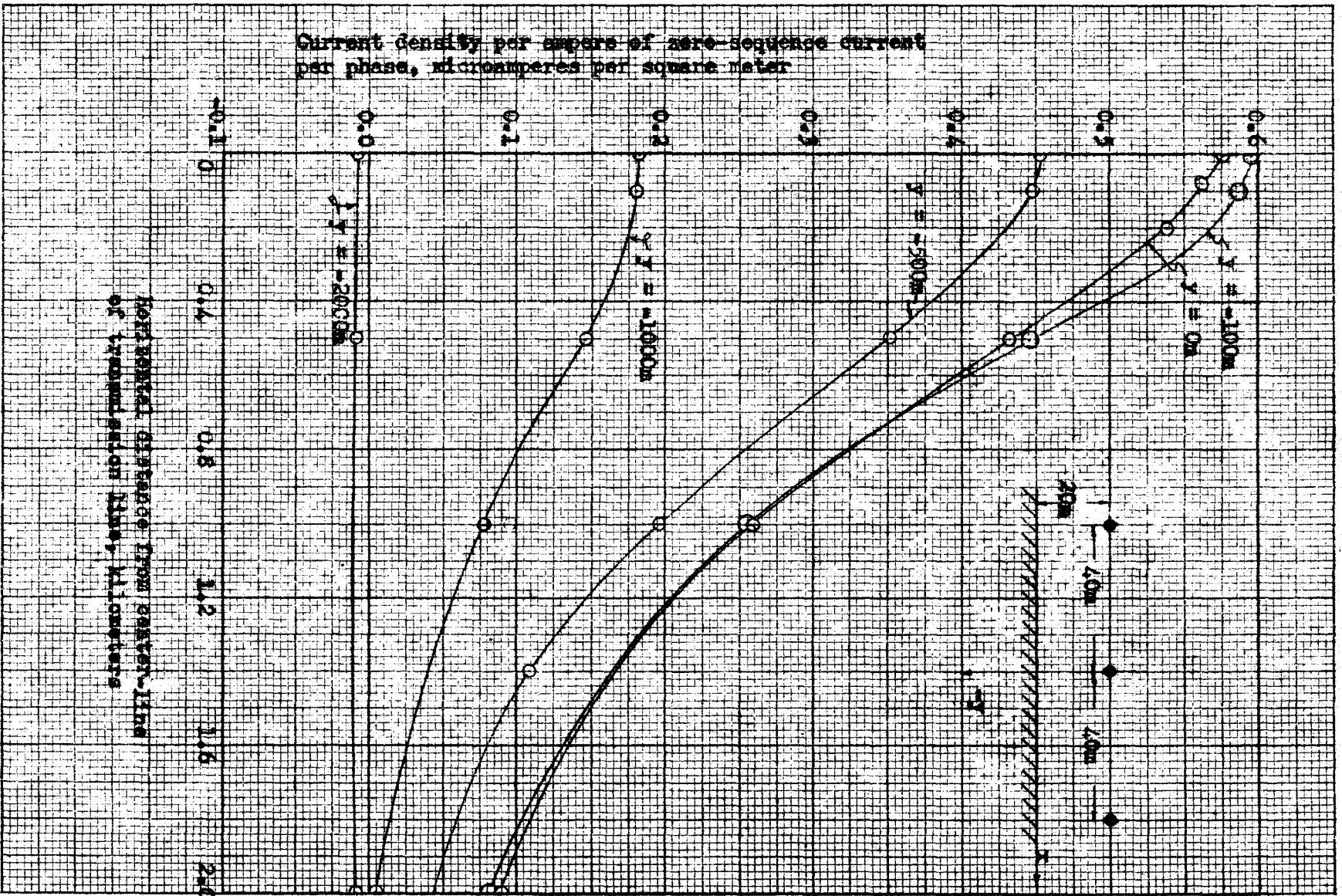
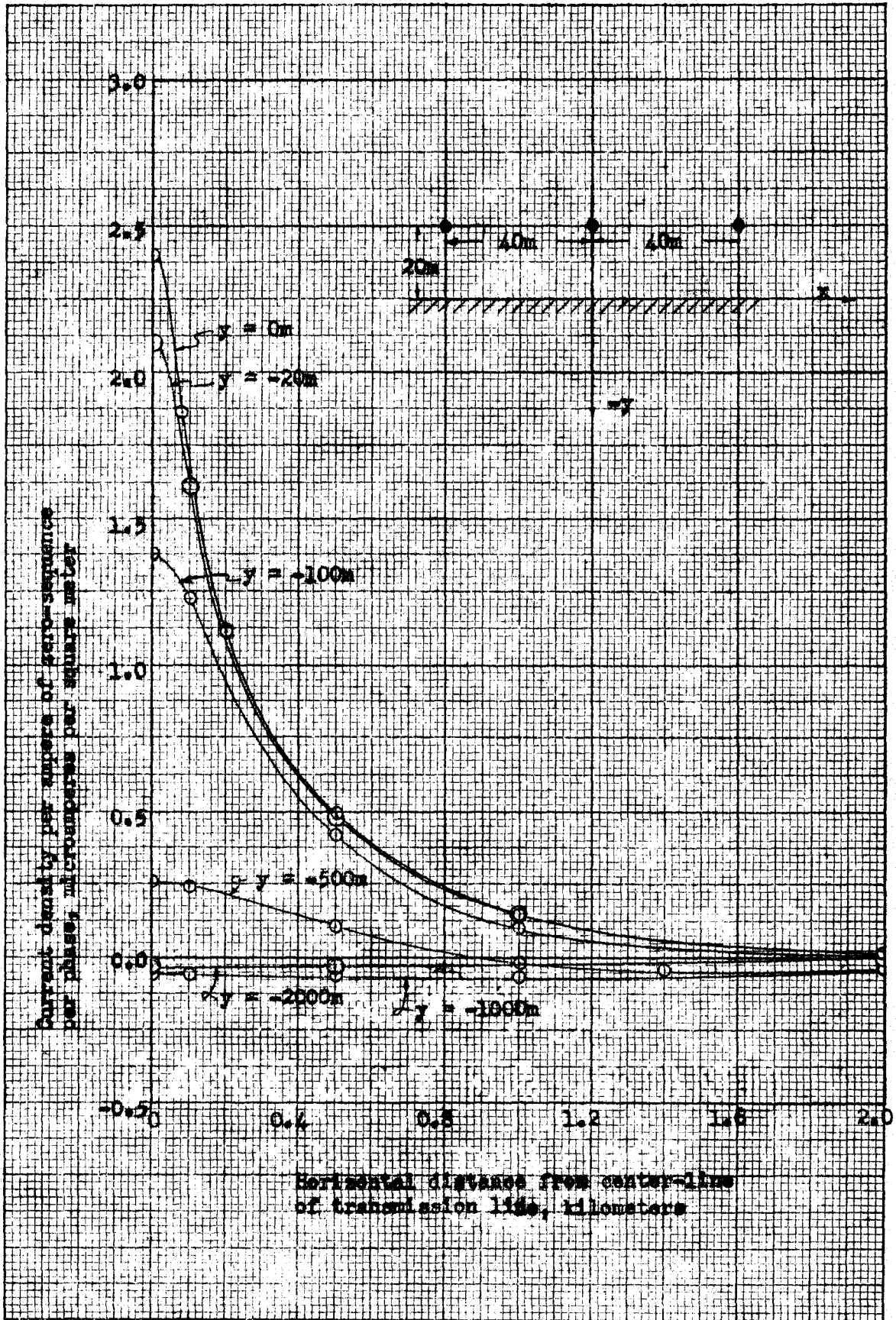


Figure 12. Imaginary component of zero-sequence current density under a three-phase transmission line with forty meter spacing and ground return



IV. IMPEDANCE OF THE EARTH

A. Resistance

1. Calculations

The impedance of the earth consists of a resistive component and a reactive component. The resistive component may be calculated by graphically integrating the I^2R losses in small elements and then dividing the total power loss by the total current squared. The details of this calculation are given below and the numerical values obtained are shown in Appendix D.

Figures 6 and 7 were used for the calculation of the resistive component of the impedance. The incremental areas used were those included between constant current density lines and were measured by means of a planimeter. Because of the conical shapes of the areas, the average value of the current is not the arithmetic mean of the two extreme values. It was assumed that a reasonable value to use would lie on the constant current density line which divided the area in question into two equal parts and that the curves were conics. The value of the radius of curvature of the conic which divides the area formed by two adjacent curves into two equal areas is $r = \sqrt{\left(\frac{r_1^2 + r_2^2}{2}\right)}$ where r_1 and r_2 are the radii of curvature of the bounding curves as they intersect the y-axis. The y-axis is used

for this calculation because of the uniformity of the constant current density lines as they intersect this axis.

As an example, the power loss in the area bounded by the 0.2 and 0.3 $\mu\text{amp}/\text{m}^2$ curves were calculated from Figure 6. The area as measured with the planimeter was 0.342 km^2 . The calculated mean radius from Figure 8 was $r = \sqrt{\frac{(0.95^2 + (0.75)^2)}{2}} = 0.86 \text{ km}$. From a plot of the real current density vs distance below the surface of the earth at $x = 0$, the value of current density for 0.86 km below the surface was found to be 0.245 $\mu\text{amp}/\text{m}/\text{phase}$. The power loss was therefore $[0.245(10^{-6})(3)(0.342)(10^6)(2)]^2 \left[\frac{100}{2(342)(10^6)} \right] = 36.9$ $\mu\text{watts}/\text{m}$. Summing the losses in each area and computing the resistance per phase from the equation

$$r_e = \frac{P}{3I^2} \quad (64)$$

gave a resistance per phase of 0.270 ohm/mile.

2. Discussion

The errors involved in this calculation arose from the assumption that all of the current flowing in the earth was confined to a finite region and that the average value of current in an incremental area could be used in computing the power loss in that area. It should also be noted here that the sum of the incremental currents was 0.5 percent low. This was due to errors in assigning values of average current density. The power loss calculated using the low values of current could therefore be about one percent low.

The calculated value of resistance compares favorably with the 0.272 ohm/mile obtained from Clarke's chart¹⁶ and if the one percent error discussed above is included, the values are practically equal.

B. Reactance

1. Calculations

The reactance was calculated using the assumption that the earth may be divided into a finite number of conductors of circular cross-section with each conductor carrying a particular portion of the current. For this calculation, only the real component of current was considered because the imaginary flux linkages are zero. Figure 6 was used to determine the currents and distances to be used in the equation for the reactance and was divided into the assumed conductors by drawing circles between current density lines tangent to the curves and the adjacent circles. The current carried by each conductor was determined by drawing a line from one current density line to the next through the points of tangency of the adjacent circles, measuring this area with a planimeter, and calculating the current as in Part IV, A, 1. Two exceptions to this procedure were made. The 0.05 $\mu\text{amp}/\text{m}^2$ curve was not used and the area below the 0.00 curve was divided by approximating the center of a tetragon. A tabulation of the calculations is shown in Appendix D.

The equation¹⁷ for the reactance due to the flux linkages one foot from the wire is

$$X_e = \frac{3.218(10^{-4})\omega}{I} (I_1 \ln \frac{1}{D_{o1}} + I_2 \ln \frac{I}{D_{og}} + \dots + I_n \ln \frac{I}{D_{on}})$$

(65)

where

I = total current flowing in the earth, amps

I_n = current flowing in the assumed conductors, amps

D_{on} = distance from the transmission line conductor

to the center of the assumed conductor, ft

X_e = reactance of the earth, ohms/mile of transmission line.

Substitution of the various computed values into equation (65)

gave $X_e = 2.84$ ohms/mile.

2. Discussion

The errors involved in this calculation were similar to those described in Part IV, A, 2 except as they applied to the reactance calculation. Additional error in the calculation of the flux linkages was introduced by the assumption that the earth is composed of circular conductors. Since the sum of currents carried by the assumed conductors was low by about two percent, the computed value of reactance should be low.

An approximate calculation was made to determine the magnitude of the flux linkages contributed by the imaginary component of current. This calculation was made to verify the statement made in Part 1 above and the value obtained using Figure 7 was less than five

percent of X_0 . A more exact calculation was not deemed necessary.

The value of X_0 obtained is approximately two percent lower than the 2.89 ohms/mile given by Westinghouse¹⁸. This error is partially due to the two percent error discussed above. Since the flux linkages between the conductors have not been considered, the computed value is not the total reactance per phase. Inclusion of the effect of these linkages reduces the above reactances to 2.00 and 2.04 ohms/mile for numerical integration and Westinghouse respectively. These latter values compare with 2.04 as obtained from Clarke's chart¹⁶.

V. SUMMARY

The equation for the zero-sequence impedance of a three-phase transmission line with ground return and the equation for the electric intensity in the earth were derived following the method of Carson². The equation for the impedance was identical with the equations of Carson¹³, however, the electric intensity equation derived included more terms than the equation of Carson¹⁰. The additional terms were due to the two additional conductors in the three-phase line.

A solution of the equation for the electric intensity could not be found and numerical solutions for three particular transmission line configurations were made. The results of these calculations are shown in Figures 2 - 12 inclusive.

Numerical integrations were made of the power losses and the flux linkages established by the current flow in the earth. From these quantities the impedance was calculated for one of the line configurations. Comparison of this impedance with values obtained from Clarke¹⁶ and Westinghouse¹⁸ are as follows:

Numerical Integration	Clarke	Westinghouse
0.272 + j2.00	0.272 + j2.04	0.272 + j2.04.

The above comparison of impedances indicates that the reactance obtained by numerical integration is about two percent lower than that obtained by Clarke and Westinghouse.

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III. APPENDICES

APPENDIX A. CARSON'S SOLUTION IN MKS UNITS

Carson's equations¹³ (27) and (28) may be written as

$$Z' = \frac{\cos t}{\pi} \int_0^{\infty} (\sqrt{v^2 + j} - v) e^{-2Dv} dv \quad (A1)$$

and

$$Z'_{12} = \frac{\cos t}{\pi} \int_0^{\infty} (\sqrt{v^2 + j} - v) e^{-(D_a + D_b)v} \cos(Sv) dv. \quad (A2)$$

Both an exact and an approximate solution to the above equations are given in reference 19. The approximate solution will be re-written here in MKS units and new nomenclature. Table 1 shows the relationship between the symbols used in this thesis and those used by Carson¹⁹ and by Clarke⁴.

Table 1. Comparison of Nomenclature

Thesis	Carson	Clarke
<u>Self Impedance</u>		
$2D = 2ah$	$r = \sqrt{p^2 + q^2} = 2\sqrt{ah}$	$k = 4\pi h_a \sqrt{2\lambda f}$
$2D$	$r \cos \theta$	$k \cos \theta$
$4D^2$	$r^2 \cos 2\theta$	$k^2 \cos 2\theta$
0	$r^2 \theta \sin 2\theta$	$k^2 \theta \sin 2\theta$
<u>Mutual Impedance</u>		
$D_a + D_b = a(h_a + h_b)$	$p = \sqrt{a}(h_a + h_b)$	none
$S = as$	$q = x' = ax$	none
$\sqrt{(D_a + D_b)^2 + S^2}$	$r = \sqrt{p^2 + q^2}$	$k = 2\pi S_{ab} \sqrt{2\lambda f}$
$D_a + D_b$	$r \cos \theta$	$k \cos \theta$
$(D_a + D_b)^2 - S^2$	$r^2 \cos 2\theta$	$k^2 \cos 2\theta$
$2S(D_a + D_b) \arctan \frac{S}{D_a + D_b}$	$r^2 \theta \sin 2\theta$	$k^2 \theta \sin 2\theta$

Equations (A1) and (A2) may be written

$$Z' = \frac{\omega \mu}{\pi} \left[\frac{\pi}{8} - \frac{\sqrt{2}}{3} D + \frac{D^2}{4} (0.6728 - \ln D) - j(0.0386 + \frac{1}{2} \ln D - \frac{\sqrt{2}}{3} D) \right] \quad (A3)$$

and

$$Z'_{12} = \frac{\omega \mu}{\pi} \left\{ \frac{\pi}{8} - \frac{D_a + D_b}{3\sqrt{2}} + \frac{1}{16} \left[\left\{ (D_a + D_b)^2 - S^2 \right\} (0.6728 - \ln \frac{\sqrt{(D_a + D_b)^2 + S^2}}{2}) + 2S(D_a + D_b) \arctan \frac{S}{D_a + D_b} \right] - j \left[0.0386 + \frac{1}{2} \ln \frac{\sqrt{(D_a + D_b)^2 + S^2}}{2} - \frac{D_a + D_b}{3\sqrt{2}} \right] \right\} \quad (A4)$$

Equations (A3) and (A4) are the approximate solutions which will give answers which are less than one percent in error when applied to power transmission lines.

The solutions of equations (A3) and (A4) for a three-phase transmission line with $s = 10$ meters, $h_a = h_b = 20$ meters, $\omega = 377$, and $\alpha = \sqrt{\omega \mu \sigma} = 0.002176$ are

$$Z' = 0.09073 + j 0.3760 \text{ ohms/mile}$$

and

$$Z'_{12} = 0.9071 + j 0.3736 \text{ ohms/mile}$$

for the center to outer conductor mutual impedance.

The remaining part of the impedance including all flux linkages one foot or greater from the conductor in question may be calculated using the final term of equation (54). The value obtained for this portion is 0.9351 ohms/mile. The total impedance of the center conductor of the transmission line without transposing is $0.272 + j 2.058$ ohms/mile.

APPENDIX B. RUDEBERG'S SOLUTION

Rudenberg²⁰ formulated a solution for the earth current density by assuming that the conductor was located at the center of a semi-circular trough in the surface of the earth with the radius of the trough equal to the height of the conductor above the earth. The equation for current density (equation 17 of reference 20) in MKS units is

$$i = \sqrt{j} \frac{dH_0(\sqrt{j} \frac{a}{\sqrt{2}})}{\sqrt{2} \pi H_1(\sqrt{j} \frac{a}{\sqrt{2}})} I e^{-j\omega t} \quad (B1)$$

and the approximate solution applicable to power transmission lines is (equation 20 of reference 20)

$$i = -\frac{a^2}{4} H_0(\sqrt{j} \frac{a}{\sqrt{2}}) I e^{-j\omega t} \quad (B2)$$

The current density at the surface of the earth, where $\rho = h = 20$ meters and with $\omega = 377$ radians/sec, is approximately

$$i = -(0.5 \pm j2.4) I e^{-j\omega t} \quad (B3)$$

The sign of the reactive term depends on the Hankel function²¹. Rudenberg used the negative sign and obtained an earth current density with a lagging component. The results shown in Figure 8 of this thesis do not verify this.

The current density at the surface of the earth near the transmission line has a leading phase angle, and therefore the Hankel function $H^{(2)}$ should be used in equations (B1) and (B2).

Woodruff²² derived the equation for the current density inside a circular conductor and made plots of the instantaneous current density along a diameter of an isolated round wire. These indicate that the imaginary component of current density at the surface leads the imaginary component at any point inside the wire. Although Woodruff's problem was quite different from Rudenberg's, the solution does indicate that the imaginary component of current density at the surface of the earth may be leading the imaginary component at points inside the earth.

Comparison of the values of the real components of current density indicates that the value obtained by Rudenberg is about 17 percent low at $x = y = 0$ and is zero at $\rho = 2600$ meters, whereas in Figure 6 the density is zero at $x = 0$, $y = -1800$ meters and at $y = 0$, $x = \infty$. Comparison of the magnitudes of the imaginary components is not possible but the distances to the points where these values are zero are $\rho = 1100$ for Rudenberg's solution, $x = 0$, $y = -850$ and $y = 0$, $x = 2500$ meters obtained from Figure 7.

According to Clarke²³, the reactance term of Rudenberg's is slightly lower than that of Carson's but this is not obvious from the current density plots. No attempt was made to verify this difference but the discussion below indicates that the value obtained by Rudenberg was not based on sound principles.

Substitution of the Hankel function $H^{(2)}$ into Rudenberg's equations (23) and (27) yields

$$H_0^{(2)}\left(\sqrt{j} \frac{a\rho}{\sqrt{2}}\right) = \frac{1}{2} + j \frac{2}{\pi} \ln \frac{2\sqrt{2}}{\gamma a \rho} \quad (\text{B4})$$

and

$$V = -z\omega I(10^{-7})\left(\frac{\pi}{2} + j \frac{2}{\pi} \ln \frac{2\sqrt{2}}{\gamma a \rho}\right) . \quad (\text{B5})$$

Equation (B5) represents the voltage drop at the surface of the earth for z meters of transmission line. This drop is equal in magnitude to the voltage drop between any two points z meters apart where z is measured parallel to the conductor, however, the phase angle varies with distance from the conductor. The phase angle associated with the current density (or electric intensity) at the surface of the earth directly beneath the line is not necessarily the phase angle associated with the total impedance of the earth. In fact, the reactance associated with the earth is a function of the real component of current density rather than the reactive component because the sum of the flux linkages with the reactive component is zero.

To integrate the flux linkages and obtain the inductance, the total current located in the region extending from h to ρ must be obtained by integrating equation (B1).

$$\int_h^\infty i\pi\rho d\rho = \int_h^\infty \sqrt{j} \frac{\rho H_0(\sqrt{j} \frac{a\rho}{\sqrt{2}})}{H_1(\sqrt{j} \frac{a\rho}{\sqrt{2}})} I e^{-j\omega t} d\rho$$

$$= \left(\frac{\rho H_1(\sqrt{j} \frac{a\rho}{\sqrt{2}})}{h H_1(\sqrt{j} \frac{a\rho}{\sqrt{2}})} - 1 \right) . \quad (B6)$$

The differential flux linkages may be written as

$$d\lambda = \frac{\mu}{2\pi\rho} \left[1 + \frac{\rho H_1(\sqrt{j} \frac{a\rho}{\sqrt{2}})}{h H_1(\sqrt{j} \frac{a\rho}{\sqrt{2}})} - 1 \right]^2 I e^{-j\omega t} d\rho$$

$$= \frac{\mu\rho}{2\pi} \left[\frac{H_1(\sqrt{j} \frac{a\rho}{\sqrt{2}})}{h H_1(\sqrt{j} \frac{a\rho}{\sqrt{2}})} \right]^2 I e^{-j\omega t} d\rho . \quad (B7)$$

Since the flux linkages between the conductor current I and the imaginary component of current density integrate to zero, only the real part of equation (B7) contributes to the differential flux linkages. The inductance may be obtained by integrating equation (B7) between the limits of h and ∞ with the following result²⁴

$$L = \frac{\mu\rho^2}{4\pi h^2} \left\{ \frac{\left[H_1(\sqrt{j} \frac{a\rho}{\sqrt{2}}) \right]^2 - H_0(\sqrt{j} \frac{a\rho}{\sqrt{2}}) H_2(\sqrt{j} \frac{a\rho}{\sqrt{2}})}{\left[H_1(\sqrt{j} \frac{a\rho}{\sqrt{2}}) \right]^2} \right\} \Bigg|_h^\infty \quad (B8)$$

$$= \frac{\mu}{4\pi} \left[\frac{H_0(\sqrt{j} \frac{ah}{\sqrt{2}}) H_2(\sqrt{j} \frac{ah}{\sqrt{2}})}{[H_1(\sqrt{j} \frac{ah}{\sqrt{2}})]^2} - 1 \right] . \quad (B9)$$

Now ²⁵

$$H_2(\sqrt{j} \frac{ah}{\sqrt{2}}) = \frac{2\sqrt{2}}{\sqrt{j}ah} H_1(\sqrt{j} \frac{ah}{\sqrt{2}}) - H_0(\sqrt{j} \frac{ah}{\sqrt{2}}) \quad (B10)$$

and therefore

$$L = \frac{\mu}{4\pi} \left\{ \frac{H_0(\sqrt{j} \frac{ah}{\sqrt{2}})}{H_1(\sqrt{j} \frac{ah}{\sqrt{2}})} \left[\frac{2\sqrt{2}}{\sqrt{j}ah} - \frac{H_0(\sqrt{j} \frac{ah}{\sqrt{2}})}{H_1(\sqrt{j} \frac{ah}{\sqrt{2}})} \right] - 1 \right\} . \quad (B11)$$

For a single wire line with $h = 20$ meters, $L = 2.81$ millihenries/mile. The inductance for this line as calculated using Carson's equations is 2.56 millihenries/mile, a slightly smaller value. This is not in accord with the statement by Clarke²³ but should not be unexpected because the current distribution in the earth using Rudenberg's equation is such that the real component at the surface of the earth is slightly small and does not decay quite so rapidly as the distribution shown in Figure 6.

APPENDIX C. DATA AS CALCULATED BY THE IBM MACHINE

Spacing Meters	Horizontal Distance Meters	Vertical Distance Meters	Imaginary Component $\mu\text{amp}/\text{m}^2$	Real Component $\mu\text{amp}/\text{m}^2$	Percent Error Imaginary	Percent Error Real	Δu	u
10	0	0	2.691	0.569	-0.09	4.22	0.8	27.2
"	20	"	2.538	0.574	-0.00	2.28	0.6	19.2
"	100	"	1.572	0.564	-0.02	2.34	0.6	21.6
"	500	"	0.506	0.437	-0.04	2.21	0.5	19.0
"	1000	"	0.136	0.259	-1.01	4.90	0.5	22.0
"	2000	"	0.011	0.094	1.81	0.88	0.1	10.2
"	3000	"	-0.006	0.037		1.05	0.1	10.4
"	4000	"	0.003	0.020	-3.33	1.24	0.08	10.4
20	0	0	2.565	0.569	-0.08	4.23	0.8	19.2
"	40	"	2.314	0.567	-0.12	4.25	0.8	19.2
"	100	"	1.588	0.558	-0.17	4.33	0.8	24.0
"	500	"	0.495	0.440	-0.09	0.72	0.3	14.4
"	1000	"	0.144	0.259	-0.66	4.90	0.5	21.0
"	2000	"	0.008	0.084	-2.84	5.95	0.3	18.0
40	0	0	2.413	0.571	-0.32	6.34	1.0	20.0
"	28	"	2.337	0.568	-0.11	4.25	0.8	19.2
"	80	"	1.858	0.561	-0.16	4.30	0.8	17.6
"	200	"	1.111	0.539	-0.00	2.48	0.6	14.4
"	500	"	0.490	0.432	-0.20	3.43	0.6	24.0
"	1000	"	0.140	0.259	-0.79	4.90	0.5	22.0
"	2000	"	0.013	0.090	-0.37	1.87	0.2	12.0

APPENDIX C (continued)

Spacing Meters	Horizontal Distance Meters	Vertical Distance Meters	Imaginary Component $\mu\text{amp}/\text{m}^2$	Real Component $\mu\text{amp}/\text{m}^2$	Percent Error Imaginary	Percent Error Real	Δu	u
10	0	20	2.265	0.581	-0.08	4.07	0.8	27.2
"	20	"	2.193	0.580	-0.08	4.07	0.8	20.8
"	100	"	1.548	0.569	-0.12	4.18	0.8	27.2
"	500	"	0.473	0.441	0.03	2.13	0.5	22.0
"	1000	"	0.134	0.259	-0.42	4.80	0.5	21.0
20	0	20	2.217	0.581	-0.08	4.07	0.8	20.8
"	20	"	2.180	0.580	-0.08	4.07	0.8	25.6
"	100	"	1.561	0.577	0.02	1.48	0.5	20.0
"	500	"	0.474	0.436	-0.08	3.32	0.6	22.8
"	1000	"	0.129	0.259	-0.42	4.80	0.5	17.0
40	0	20	2.106	0.581	-0.32	6.16	1.0	20.0
"	20	"	2.076	0.581	-0.32	6.16	1.0	24.0
"	100	"	1.610	0.574	-0.01	2.24	0.6	14.4
"	500	"	0.479	0.444	0.03	1.27	0.4	17.6
"	1000	"	0.134	0.268	0.10	2.51	0.4	20.0

APPENDIX C (continued)

Spacing Meters	Horizontal Distance Meters	Vertical Distance Meters	Imaginary Component jamp/m ²	Real Component jamp/m ²	Percent Error Imaginary	Percent Error Real	Δu	u
10	0	100	1.407	0.597	-0.01	4.64	0.9	14.4
"	20	"	1.398	0.596	-0.01	4.65	0.9	14.4
"	100	"	1.236	0.592	+0.13	2.77	0.7	12.6
"	500	"	0.416	0.450	+0.30	1.86	0.5	15.0
"	1000	"	0.099	0.268	+0.40	1.10	0.3	13.8
"	2000	"	-0.007	0.082	0.86	0.38	0.1	10.2
"	3000	"	-0.004	0.032	-1.67	1.04	0.1	8.8
"	4000	"	-0.002	0.017	6.00	2.11	0.1	5.4
20	0	100	1.390	0.596	-0.16	5.65	1.0	14.0
"	20	"	1.381	0.596	-0.16	5.66	1.0	14.0
"	100	"	1.206	0.585	-0.21	5.80	1.0	14.0
"	500	"	0.415	0.445	0.33	2.95	0.6	16.6
"	1000	"	0.102	0.268	0.14	0.46	0.2	9.4
"	2000	"	-0.008	0.081	-2.32	1.77	0.2	16.8
40	0	100	1.374	0.595	-0.16	5.67	1.0	16.0
"	40	"	1.342	0.593	-0.17	5.69	1.0	16.0
"	100	"	1.225	0.587	0.08	3.74	0.8	12.8
"	500	"	0.417	0.445	0.32	2.95	0.6	13.2
"	1000	"	0.100	0.255	0.90	4.44	0.5	13.0
"	2000	"	-0.007	0.081	2.15	1.77	0.2	12.4

APPENDIX C (continued)

Spacing Meters	Horizontal Distance Meters	Vertical Distance Meters	Imaginary Component $\mu\text{amp}/\text{m}^2$	Real Component $\mu\text{amp}/\text{m}^2$	Percent Error Imaginary	Percent Error Real	Δu	u
10	0	500	0.256	0.453	1.32	0.64	0.5	7.0
"	100	"	0.249	0.448	1.36	0.65	0.5	7.0
"	500	"	0.101	0.352	1.01	0.31	0.3	4.2
"	1000	"	-0.024	0.196	-1.95	0.25	0.2	5.6
"	2000	"	-0.038	0.042	-0.06	0.07	0.05	
"	3000	"	-0.015	0.012	-0.82	1.11	0.10	6.4
"	4000	"	-0.007	0.007	-0.53	0.44	0.05	3.6
20	0	500	0.256	0.453	0.74	0.39	0.4	5.6
"	100	"	0.246	0.448	0.77	0.40	0.4	5.6
"	500	"	0.103	0.352	2.02	0.54	0.4	7.2
"	1000	"	-0.023	0.196	-0.45	0.06	0.1	4.0
"	2000	"	-0.036	0.042	-0.35	0.29	0.1	3.2
40	0	500	0.255	0.452	0.77	0.40	0.4	6.4
"	100	"	0.246	0.447	0.77	0.40	0.4	6.4
"	500	"	0.103	0.352	2.02	0.54	0.4	6.4
"	1000	"	-0.024	0.196	-1.96	0.25	0.2	5.2
"	1400	"	-0.047	0.109	-1.02	0.48	0.2	5.2
"	2800	"	-0.017	0.014	-3.87	5.08	0.2	5.6
"	4000	"	-0.007	0.007	-2.00	2.17	0.1	5.2

APPENDIX C (continued)

Spacing Meters	Horizontal Distance Meters	Vertical Distance Meters	Imaginary Component $\mu\text{amp}/\text{m}^2$	Real Component $\mu\text{amp}/\text{m}^2$	Percent Error Imaginary	Percent Error Real	Δu	u
10	0	1000	-0.050	0.182	-1.34	0.01	0.3	4.8
"	100	"	-0.051	0.180	-1.32	0.01	0.3	4.2
"	500	"	-0.063	0.145	-1.10	0.03	0.3	4.2
"	1000	"	-0.070	0.077	-0.45	1.84	0.2	4.4
"	2000	"	-0.034	0.005	-0.56	0.02	0.05	2.6
"	3000	"	-0.010	-0.002	-0.80	-0.22	0.1	4.2
"	4000	"	-0.004	0.000	-2.06	-1.37	0.1	3.8
20	0	1000	-0.050	0.182	-0.60	0.01	0.2	4.8
"	100	"	-0.051	0.180	-0.59	0.01	0.2	4.8
"	500	"	-0.063	0.145	-0.48	0.01	0.2	3.6
"	1000	"	-0.067	0.078	-2.97	-0.23	0.4	6.8
"	2000	"	-0.034	0.005	-1.09	0.32	0.2	6.0
"	4000	"	-0.004	0.000	-0.39	-0.54	0.05	2.6
40	0	1000	-0.050	0.182	-0.94	0.01	0.25	4.0
"	100	"	-0.051	0.180	-0.93	0.01	0.25	4.0
"	500	"	-0.063	0.145	-0.76	0.01	0.25	5.0
"	1000	"	-0.070	0.077	-0.73	0.02	0.25	4.0
"	1400	"	-0.058	0.035	-0.93	0.01	0.25	4.0
"	2000	"	-0.033	0.005	-1.96	-0.97	0.25	4.0
"	2800	"	-0.011	-0.002	-10.43	11.83	0.25	4.0
"	4000	"	-0.004	0.000	-0.44	-0.34	0.05	3.1

APPENDIX C (continued)

Spacing Meters	Horizontal Distance Meters	Vertical Distance Meters	Imaginary Component $\mu\text{amp}/\text{m}^2$	Real Component $\mu\text{amp}/\text{m}^2$	Percent Error		Δu	u
					Imaginary	Real		
10	0	2000	-0.030	-0.008	-0.00	0.20	0.1	2.5
"	500	"	-0.027	-0.009	-0.00	0.17	0.1	1.6
"	1000	"	-0.019	-0.012	-0.01	0.12	0.1	1.6
"	2000	"	-0.004	-0.010	-0.00	0.19	0.1	1.6
"	3000	"	0.001	-0.004	0.22	0.50	0.1	2.6
"	4000	"	0.000	-0.001	0.08	0.36	0.05	1.7
20	0	2000	-0.030	-0.008	-0.01	0.85	0.2	4.2
"	500	"	-0.027	-0.009	-0.01	0.71	0.2	4.4
"	1000	"	-0.019	-0.012	-0.01	0.14	0.1	2.6
"	2000	"	-0.004	-0.009	-0.01	0.04	0.05	1.8
"	4000	"	0.000	-0.001	0.10	0.37	0.05	2.0
40	0	2000	-0.030	-0.008	-0.01	0.85	0.2	3.6
"	500	"	-0.027	-0.009	-0.02	0.72	0.2	4.0
"	2000	"	-0.004	-0.010	-0.04	1.76	0.1	2.4
"	4000	"	0.000	-0.001	0.41	1.78	0.1	4.1

APPENDIX D. GRAPHICAL INTEGRATION

1. Resistance

Area km ²	Average Current Density $\mu\text{amp}/\text{m}^2$	*Total Current amps	**Total Resistance μ ohms	Total Power μ watts/m
	Real Component			
0.03500	0.59	0.0207	1428.5	22.06
0.08625	0.535	0.0461	579.8	44.36
0.16175	0.440	0.0712	309.1	56.41
0.23925	0.330	0.0790	209.0	46.96
0.34175	0.245	0.0837	146.3	36.90
0.72000	0.155	0.1116	69.4	31.13
0.67075	0.080	0.0537	74.5	7.73
0.66175	0.041	0.0271	75.6	2.00
1.95500	0.012	0.0235	25.6	0.50
1.27750	-0.0012	-0.0015	39.1	negligible
1.45000	-0.0038	-0.0055		"
2.42875	-0.0050	-0.0121		"
<hr/>		<hr/>		<hr/>
10.02775		0.4975		248.05

*Total current is the total current as obtained from the curves. This is only 1/2 the true current per phase.

**Total resistance is the computed resistance of the earth using twice the area given in this tabulation.

APPENDIX D (continued)

Area km ²	Average Current Density $\mu\text{amp}/\text{m}^2$	*Total Current amps	**Total Resistance μ ohms	Total Power μ watts/m
0.02825	1.25	0.0353	1770.0	79.40
0.13750	0.67	0.0921	363.6	111.0
0.13175	0.35	0.0461	379.5	29.0
0.21000	0.165	0.0347	238.1	10.3
0.41750	0.071	0.0296	119.8	3.8
3.81250	-0.005	-0.0191	13.1	0.2
1.78500	-0.015	-0.0268	28.0	0.7
2.33500	-0.045	-0.1051	21.4	8.5
1.15000	-0.075	-0.0863	43.5	11.7
0.10750	-0.010	-0.0011		negligible
10.11500		-0.0006		254.6

Grand Total
Power 502.7 μ watts/m.

$$R_0 = \frac{502.7 \times 1609 \times 10^{-6}}{3}$$

= 0.270 ohms/mile.

*Total current is the total current as obtained from the curves. This is only 1/2 the true current per phase.

**Total resistance is the computed resistance of the earth using twice the area given in this tabulation.

APPENDIX D (continued)

2. Reactance

Area km ²	Current Density amp/m ²	Total Current amps (I)	Distance from Center Cond. Pt. (D)	$-I \times \frac{1}{n} \frac{1}{D}$	Reactance ohms/ml.
0.03500	0.590	0.0207	525	0.12965	0.09437
0.02125	0.538	0.0114	1214	0.08097	0.05894
0.02625	"	0.0141	1099	0.09874	0.07187
0.03875	"	0.0208	919	0.14192	0.10330
0.02750	0.440	0.0121	1706	0.09008	0.06557
0.02000	"	0.0088	1690	0.06541	0.04761
0.04750	"	0.0209	1710	0.15558	0.11325
0.06500	"	0.0286	1660	0.21206	0.15436
0.02625	0.330	0.0087	2200	0.06696	0.04874
0.02750	"	0.0091	2210	0.07008	0.05101
0.02750	"	0.0091	2280	0.07036	0.05122
0.03875	"	0.0128	2390	0.09957	0.07248
0.06375	"	0.0210	2490	0.16422	0.11954
0.05375	"	0.0177	2490	0.13941	0.10075
0.05000	0.245	0.0125	2810	0.09767	0.07109
0.04625	"	0.0113	2840	0.08985	0.06540
0.04125	"	0.0101	2920	0.08059	0.05866
0.04625	"	0.0113	3120	0.09092	0.06618
0.05125	"	0.0126	3280	0.10200	0.07425
0.11000	"	0.0270	3440	0.21987	0.16004
0.08000	0.165	0.0124	3610	0.10157	0.07393
0.08125	"	0.0126	3610	0.10321	0.07513
0.09000	"	0.0140	3740	0.11518	0.08394
0.10250	"	0.0159	4050	0.13207	0.09613
0.14375	"	0.0223	4400	0.18708	0.13618
0.22125	"	0.0343	4820	0.29088	0.21173

APPENDIX D (continued)

Area km ²	Current Density μamp/m ²	Total Current amps (I)	Distance from Center Cond. ft. (D)	$-I / n D$	Reactance ohms/mi.
0.10250	0.060	0.0062	4630	0.05233	0.03809
0.12500	"	0.0075	4590	0.06324	0.04603
0.13125	"	0.0079	4760	0.06690	0.04870
0.14125	"	0.0085	5020	0.07243	0.05272
0.15500	"	0.0093	5530	0.08015	0.05834
0.25250	"	0.0152	6200	0.13273	0.09661
0.43125	"	0.0259	7180	0.22997	0.16740
0.06625	0.012	0.0008	5590	0.00690	0.00502
0.06500	"	0.0008	5660	0.00691	0.00503
0.07750	"	0.0009	5740	0.00779	0.00567
0.08125	"	0.0010	5840	0.00867	0.00631
0.07500	"	0.0009	6020	0.00783	0.00570
0.08500	"	0.0010	6300	0.00745	0.00542
0.91250	"	0.0011	6630	0.00968	0.00705
0.10875	"	0.0013	7180	0.01154	0.00840
0.17750	"	0.0021	7910	0.01885	0.01372
0.29000	"	0.0035	9020	0.03188	0.02321
0.37875	"	0.0046	10500	0.04259	0.03100
0.45000	"	0.0054	12300	0.05085	0.03701
0.73000	-0.003	-0.0022	7220	-0.01955	-0.01423
1.10000	-0.005	-0.0055	8530	-0.04978	-0.03623
0.81250	"	-0.0041	9510	-0.03756	-0.02734
0.75250	-0.006	-0.0045	10830	-0.04179	-0.03042
0.93500	-0.003	-0.0028	12140	-0.02632	-0.01916
0.75000	"	-0.0023	13450	-0.02187	-0.01592
10.76875		0.4894			2.8437